Naturality for Free

The category interpretation of directed type theory

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Types are abstract

This has powerful consequences:

Parametricity
: *Polymorphic functions preserve all logical relations.*

Univalence
: *Isomorphic types are equal.*

How are these related?

List reversal is natural

$$\mathsf{List} : \mathsf{Set} \to \mathsf{Set}$$

$$\mathsf{rev} : \prod_{A:\mathsf{Set}} \mathsf{List}\,A \to \mathsf{List}\,A$$

$$f : A \to B$$

$$\begin{tikzcd}
\mathsf{List}\,A \arrow[r, "\mathsf{rev}_A"] \arrow[d, "\mathsf{List}\,f"'] & \mathsf{List}\,A \arrow[d, "\mathsf{List}\,f"] \\
\mathsf{List}\,B \arrow[r, "\mathsf{rev}_B"'] & \mathsf{List}\,B
\end{tikzcd}$$

$$\mathsf{List}\ f \circ \mathsf{rev}_A = \mathsf{rev}_B \circ \mathsf{List}\ f$$

Proof by induction

$$\mathsf{List}\, f\, [a_0, a_1, \ldots, a_{n-1}] = [f\, a_0, f\, a_1, \ldots, f\, a_{n-1}]$$

$$\mathsf{rev}_A\, [a_0, a_1, \ldots, a_{n-1}] = [a_{n-1}, \ldots, a_1, a_0]$$

$$\begin{aligned}
(\mathsf{rev}_B \circ \mathsf{List}\, f)\, [a_0, a_1, \ldots, a_{n-1}] &= \mathsf{rev}_B\, (\mathsf{List}\, f\, [a_0, a_1, \ldots, a_{n-1}]) \\
&= \mathsf{rev}_B\, [f\, a_0, f\, a_1, \ldots, f\, a_{n-1}]) \\
&= [f\, a_{n-1}, \ldots, f\, a_1, f\, a_0]) \\
&= \mathsf{List}\, f\, [a_{n-1}, \ldots, a_1, a_0] \\
&= \mathsf{List}\, f\, (\mathsf{rev}_A\, [a_0, a_1, \ldots, a_{n-1}]) \\
&= (\mathsf{List}\, f \circ \mathsf{rev}_A)\, [a_0, a_1, \ldots, a_{n-1}]
\end{aligned}$$

Everything is natural . . .

$$F, G : \mathsf{Set} \to \mathsf{Set}$$
$$\alpha : \prod_{A:\mathsf{Set}} F\,A \to G\,A$$
$$f : A \to B$$

$$\begin{tikzcd}
F\,A \arrow[r, "\alpha_A"] \arrow[d, "F\,f"'] & G\,A \arrow[d, "G\,f"] \\
F\,B \arrow[r, "\alpha_B"'] & G\,B
\end{tikzcd}$$

. . . but we can't prove it.

- Naturality (parametricity) is a metatheorem, but we cannot prove it internally
- Solution: extend MLTT with constructs to internalise free theorems (Nuyts et al., Bernardy et al., ...)
- Can we link this to Univalence/HoTT?

The hint (HoTT)

$$F, G : \mathsf{Set} \to \mathsf{Set}$$
$$\alpha : \prod_{A:\mathsf{Set}} F\,A \simeq G\,A$$
$$f : A \simeq B$$

$$\begin{tikzcd} F\,A \arrow[r, "\alpha_A"] \arrow[d, "F\,f"'] & G\,A \arrow[d, "G\,f"] \\ F\,B \arrow[r, "\alpha_B"'] & G\,B \end{tikzcd}$$

- $A \simeq B$ means isomorphism (we work in a set-level setting);
- This is provable in HoTT, from Univalence + J.

- Set-level univalent type theory can be interpreted into the groupoid model (Hofmann & Streicher).
- The model validates a univalent universe of sets;
- Idea: replace groupoids with categories;
- This gives a semantics for a *directed* type theory with
  - an internal Hom type;
  - *directed Univalence*;

The category with families of categories

Contexts	$\mathsf{Con} : \mathsf{Set}$	$\Gamma : \mathsf{Con}$	$[\![\Gamma]\!] : \mathsf{Cat}$
Types	$\mathsf{Ty} : \mathsf{Con} \to \mathsf{Set}$	$A : \mathsf{Ty}\,\Gamma$	$[\![A]\!] : [\![\Gamma]\!] \to \mathsf{Cat}$
Terms	$\mathsf{Tm} : (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Set}$	$a : \mathsf{Tm}\,\Gamma\,A$	$[\![a]\!] : \mathsf{Sect}\,\widehat{[\![A]\!]}$
Substitutions	$\mathsf{Tms} : \mathsf{Con} \to \mathsf{Con} \to \mathsf{Set}$	$\gamma : \mathsf{Tms}\,\Gamma\,\Delta$	$[\![\gamma]\!] : [\![\Gamma]\!] \to [\![\Delta]\!]$

Operations on contexts

$$\frac{}{\bullet : \mathsf{Con}} \qquad \frac{A : \mathsf{Ty}\,\Gamma}{\Gamma.A : \mathsf{Con}}$$

$$[\![\bullet]\!] := \mathbf{1}$$

$$|[\![\Gamma.A]\!]| := (\gamma : |[\![\Gamma]\!]|) \times |[\![A]\!]\,\gamma|$$
$$[\![\Gamma.A]\!]((\gamma, a), (\gamma', a')) := (f : [\![\Gamma]\!](\gamma, \gamma')) \times ([\![A]\!]\,\gamma')([\![A]\!]\,f\,a, a')$$

Grothendieck construction

Opposites

$$\frac{\Gamma : \mathsf{Con}}{\Gamma^- : \mathsf{Con}} \qquad \frac{A : \mathsf{Ty}\,\Gamma}{A^- : \mathsf{Ty}\,\Gamma}$$

$$\llbracket \Gamma^- \rrbracket := \llbracket \Gamma \rrbracket^{op}$$
$$\llbracket A^- \rrbracket := \mathsf{op} \circ \llbracket A \rrbracket$$

where $\mathsf{op} : \mathsf{Cat} \longrightarrow \mathsf{Cat}$ takes $\mathcal{C}$ into $\mathcal{C}^{op}$.

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- But what is $(\Gamma.A)^-$?
- $\Gamma^-.A^-$ doesn't typecheck.

Contravariant context extension

$$\frac{A : \mathsf{Ty}\,\Gamma^-}{\Gamma.^- A : \mathsf{Con}}$$

$$|[\![\Gamma.^- A]\!]| := (\gamma : |[\![\Gamma]\!]|) \times |[\![A]\!]\,\gamma|$$
$$[\![\Gamma.^- A]\!]((\gamma, a), (\gamma', a')) := (f : [\![\Gamma]\!](\gamma, \gamma')) \times ([\![A]\!]\,\gamma)(a, [\![A]\!]\,f\,a')$$

$$(\Gamma.A)^- = \Gamma^-.^- A^-$$

Σ-types

$$\frac{A : \mathsf{Ty}\,\Gamma \quad B : \mathsf{Ty}\,\Gamma.A}{\Sigma\, A\, B : \mathsf{Ty}\,\Gamma}$$

On objects:

$$[\![\Sigma\, A\, B]\!]\,\gamma := ([\![A]\!]\,\gamma).([\![B]\!]\,\gamma)$$

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What about $(\Sigma\,A\,B)^{-}$?

Σ^--types

$$\frac{A : \mathsf{Ty}\,\Gamma \qquad B : \mathsf{Ty}\,(\Gamma.A^-)}{\Sigma^-\,A\,B : \mathsf{Ty}\,\Gamma}$$

On objects:

$$[\![\Sigma^-\,A\,B]\!]\,\gamma := ([\![A]\!]\,\gamma).^-([\![B]\!]\,\gamma)$$

$$(\Sigma\,A\,B)^- = \Sigma^-\,A^-\,B^-$$

Σ-types with polarities

$$\frac{A : \mathsf{Ty}\ \Gamma \qquad B : \mathsf{Ty}\ (\Gamma.A^s)}{\Sigma^s\, A\, B : \mathsf{Ty}\ \Gamma} \qquad \frac{M : \mathsf{Tm}^s\ \Gamma\ A \qquad N : \mathsf{Tm}^s\ \Gamma\ B[M]}{\langle M, N\rangle^s : \mathsf{Tm}^s\ \Gamma\ (\Sigma^s\, A\, B)}$$

$$\frac{M : \mathsf{Tm}^s\ \Gamma\ (\Sigma^s\, A\, B)}{\pi_1^s\, M : \mathsf{Tm}^s\ \Gamma\ A} \qquad \frac{M : \mathsf{Tm}^s\ \Gamma\ (\Sigma^s\, A\, B)}{\pi_2^s\, M : \mathsf{Tm}^s\ \Gamma\ (B[\pi_1^s\, M])}$$

where $s = \{+, -\}$ and $\mathsf{Tm}^-\, \Gamma\, A \equiv \mathsf{Tm}\, \Gamma^-\, A^-$.

Π-types

- Groupoid construction of Π generalises to categories...
- ... but we need to be careful with polarities.

$$\frac{A : \mathsf{Ty}\,\Gamma^{-} \qquad B : \mathsf{Ty}\,(\Gamma .^{-} A)}{\Pi\, A\, B : \mathsf{Ty}\,\Gamma}$$

Π-types with polarities

$$\frac{A : \mathsf{Ty}\ \Gamma^{-} \qquad B : \mathsf{Ty}\ (\Gamma .^{-} A^{s})}{\Pi^{s}\, A\, B : \mathsf{Ty}\ \Gamma}$$

$$\frac{t : \mathsf{Tm}^{s}\ (\Gamma .^{-s} A)\ B}{\lambda^{s}\ t : \mathsf{Tm}^{s}\ \Gamma\ (\Pi^{s}\, A\, B)} \qquad \frac{t : \mathsf{Tm}^{s}\ \Gamma\ (\Pi^{s}\, A\, B)}{\mathsf{app}^{s}\, t : \mathsf{Tm}^{s}\ (\Gamma .^{-s} A)\ B}$$

$$(\Pi^{s}\, A\, B)^{-} = \Pi^{-s}\, A^{-}\ B^{-}$$

Universe of sets

$$\frac{}{\mathsf{U} : \mathsf{Ty}\,\Gamma} \qquad \frac{A : \mathsf{Tm}\ \Gamma\ \mathsf{U}^s}{\mathsf{El}\,A : \mathsf{Ty}\ \Gamma^s}$$

closed under $\Pi, \Sigma, \ldots$.

$$\begin{aligned} |[\![\mathsf{U}]\!]\,\gamma| &:= \mathsf{Set} \\ ([\![\mathsf{U}]\!]\,\gamma)(A, B) &:= A \to B \end{aligned}$$

$$\begin{aligned} |[\![\mathsf{El}\,a]\!]\,\gamma| &:= [\![a]\!]\,\gamma \\ ([\![\mathsf{El}\,a]\!]\,\gamma)(y, z) &:= (y = z) \end{aligned}$$

$$(\mathsf{El}\,A)^- = \mathsf{El}\,A$$

The Hom type

$$\frac{a : Tm\,\Gamma\,A^{-} \quad b : \mathrm{Tm}\,\Gamma\,A}{\mathrm{Hom}_A\,a\,b : \mathrm{Ty}\,\Gamma}$$

we also write $a \sqsubseteq_A b$ for $\mathrm{Hom}_A\,a\,b$.

On objects:

$$[\![\mathrm{Hom}_A\,a\,b]\!]\,\gamma := A\,\gamma(a\,\gamma, b\,\gamma)$$

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- But what about id (aka refl)?
- We would like to say

$$\frac{a : \mathrm{Tm}\,\Gamma\,A}{\mathrm{id}_a : \mathrm{Hom}_A\,a\,a}$$

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but this doesn't type check!

Core types

$$\frac{A : \mathrm{Ty}\,\Gamma}{\overline{A} : \mathrm{Ty}\,\Gamma} \qquad \frac{a : \mathrm{Tm}\,\Gamma\,\bar{A}}{a^s : \mathrm{Tm}\,\Gamma\,A^s}$$

$$[\![\overline{A}]\!] := \mathsf{core} \circ [\![A]\!]$$

$$\frac{\begin{array}{c} a, b : \mathrm{Tm}\,\Gamma\,\bar{A} \\ f : \mathrm{Tm}\,\Gamma\,(a^- \sqsubseteq_A b^+) \qquad g : \mathrm{Tm}\,\Gamma\,(b^- \sqsubseteq_A a^+) \\ l : \mathrm{Tm}\,\Gamma\,(f \circ g \sqsubseteq \mathrm{id}_b) \qquad r : \mathrm{Tm}\,\Gamma\,(g \circ f \sqsubseteq \mathrm{id}_a) \end{array}}{\overline{f, g, l, r} : \mathrm{Tm}\,\Gamma\ \ \mathrm{Hom}_{\overline{A}}\,a\,b}$$

Some issues with cores

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- Introduction rule? $A \not\longrightarrow \overline{A}$

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- Rules for core contexts? $\overline{\Gamma.A} \simeq \overline{\Gamma}.\overline{A}[...]$

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- Introduction rule? $A \not\longrightarrow \overline{A}$

$$\frac{a : \mathrm{Tm}\ \overline{\Gamma}\ A}{\overline{a} : \mathrm{Tm}\ \overline{\Gamma}\ \overline{A}}$$

- Rules for core contexts? $\overline{\Gamma.A} \simeq \overline{\Gamma}.\overline{A}[...]$
- $\overline{\overline{A}} \cong \overline{A}$

Identity morphisms

$$\frac{a : \mathsf{Tm}\,\Gamma\,\overline{A}}{\mathsf{id}_a : \mathsf{Hom}_A\, a^-\, a^+}$$

Morphism induction (J^s)

$$\frac{\begin{array}{c} a : \mathsf{Tm}\,\Gamma\,\overline{A} \\ b : \mathsf{Tm}\,\Gamma\,A \\ M : \mathsf{Ty}\,\Gamma, x : A, z : \mathsf{Hom}_{A^s}\,a^{-s}\,x \\ m : \mathsf{Tm}\,\Gamma\,M[x := a^{+}, z := \mathsf{id}_a] \\ p : \mathsf{Tm}\,\Gamma\,\mathsf{Hom}_{A^s}\,a^{-s}\,b \end{array}}{\mathsf{J}^s\,M\,m\,p : M[x := b, z := p]}$$

Morphism induction II ($\bar{\mathrm{J}}$)

$$\frac{\begin{array}{c} a : \mathrm{Tm}\ \Gamma\ \overline{A} \\ b : \mathrm{Tm}\ \Gamma\ \overline{A} \\ M : \mathrm{Ty}\ (\Gamma, x : \overline{A}, z : \mathrm{Hom}_A\, a^-\ x^+) \\ m : \mathrm{Tm}\ \Gamma\ \ M[x := a, z := \mathrm{id}_a] \\ p : \mathrm{Tm}\ \Gamma\ (\mathrm{Hom}_A\, a^-\ b^+) \end{array}}{\bar{\mathrm{J}}\, M\, m\, p : M[x := b, z := p]}$$

Directed Univalence

$$\frac{A : \mathsf{Tm}\ \Gamma\ \mathsf{U}^- \qquad B : \mathsf{Tm}\ \Gamma\ \mathsf{U}}{\mathsf{Hom}_{\mathsf{U}}\, A\, B = \mathsf{El}\,(A \to B)}$$

Directed Univalence

$$\frac{A : \mathrm{Tm}\ \Gamma\ \mathrm{U}^{-} \qquad B : \mathrm{Tm}\ \Gamma\ \mathrm{U}}{\mathrm{Hom}_{\mathrm{U}}\, A\, B = \mathrm{El}\,(A \to B)}$$

“Undirected” Univalence follows from the directed one.

$$\mathrm{Hom}_{\overline{\mathrm{U}}}\, A\, B = f : A^{-} \to B^{+}, g : B^{-} \to A^{+}, +\text{proofs}$$

Every type family is functorial

$$X : \overline{\mathrm{U}} \vdash F : \overline{\mathrm{U}}$$

For any

$$A, B : \overline{\mathrm{U}}, f : A^{-} \to B^{+}$$

We can construct

$$\mathsf{ap}\, F\, f : F[A]^{-} \to F[B]^{+}$$

from directed UA + $\overline{\mathrm{J}}$

Every polymorphic function is natural

$$X : \overline{\mathrm{U}} \vdash F, G : \overline{\mathrm{U}}$$

$$\alpha : \prod_{A : \overline{\mathrm{U}}} F[A]^{-} \to G[A]^{+}$$

For any

$$A, B : \overline{\mathrm{U}} \qquad f : A^{-} \to B^{+}$$

$$\begin{array}{ccc} F[A] & \xrightarrow{\alpha_A} & G[A] \\ {\scriptstyle \mathrm{ap}\, F\, f}\downarrow & & \downarrow{\scriptstyle \mathrm{ap}\, G\, f} \\ F[B] & \xrightarrow[\alpha_B]{} & G[B] \end{array}$$

$$\mathrm{nat}\, f : \mathrm{Hom}_{F[A]^{-} \to G[B]^{+}} (\mathrm{ap}\, G\, g \circ \alpha_A)\, (\alpha_B \circ \mathrm{ap}\, F\, f)$$

from directed UA + $\overline{\mathrm{J}}$

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$$\mathrm{nat}\, f : \mathrm{Hom}_{F[A]^{-} \to G[B]^{+}} (\mathrm{ap}\, G\, g \circ \alpha_A)\, (\alpha_B \circ \mathrm{ap}\, F\, f)$$

from directed UA + $\overline{\mathrm{J}}$ (assuming symmetric/core context)

Related work

- R. Harper & D. Licata, 2-dimensional Directed Type Theory;
- P. North, Towards a directed Homotopy Type Theory;
- A. Nuyts, MSc thesis
- E. Riehl & M. Shulman, A type theory for synthetic ∞-categories (see also Jonathan Weinberger's talk);

Future work

- Formalisation of the calculus and its semantics in Agda (ongoing);
- Researching appropriate way to represent symmetric/groupoidal/core contexts; split-context modal type theory seems relevant (see also Dan Licata's talk);
- What is the relation to logical relations?
- Can we do higher categories (full directed HoTT)?