Type-theoretic modalities for synthetic $(\infty, 1)$-categories

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Outline

Foundations: Synthetic $(\infty, 1)$-categories à la Riehl–Shulman I

In order to develop *synthetic higher category theory*, Riehl and Shulman introduced a *Simplicial Type Theory* (STT) in [RS17]: MLTT with additional layers of shapes, allowing for defining *synthetic* $(\infty, 1)$*-categories* as *complete Segal/Rezk types*.

As a main feature, STT postulates *extension types* (after Lumsdaine–Shulman), i.e. for shape inclusions $\Phi \rightarrowtail \Psi$, families $A : \Psi \to \mathcal{U}$, and partial sections $a : \prod_{t:\Phi} A(t)$ there exists the type of sections

$$\left\langle \prod_{t:\Psi} A(t) \Big|_{a}^{\Phi} \right\rangle \triangleq \left\{ \begin{array}{ccc} \Phi & \xrightarrow{a} & A \\ \downarrow & \nearrow_{\bar{a}} & \\ \Psi & & \end{array} \right\}$$

judgmentally extending a.

Example & Definition: For a type A and terms $x, y : A$, define the *hom-type* $\hom_A(x,y) := \left\langle \Delta^1 \to A \Big|_{[x,y]}^{\partial \Delta^1} \right\rangle$.

Foundations: Synthetic $(\infty, 1)$-categories à la Riehl–Shulman II

Definitions from $[\mathrm{RS17}]$:

- A type A is a *Segal type* if $(\Delta^2 \to A) \xrightarrow{\simeq} (\Lambda_1^2 \to A)$ (Joyal).
- A Segal type A is a *Rezk type* if $\mathrm{idtoiso}_A : \prod_{x,y:A} \mathrm{Id}_A(x,y) \xrightarrow{\simeq} \mathrm{iso}_A(x,y)$.
- A type A is a *discrete type* if $\mathrm{idtorarr}_A : \prod_{x,y:A} \mathrm{Id}_A(x,y) \xrightarrow{\simeq} \hom_A(x,y)$.

These notions coincide with their classical analogues in the intended semantics in (a model structure representing) the ∞-topos of simplicial spaces, $\mathrm{PSh}_\infty(\mathbb{\Delta})$.

Goal: Extend the ∞-category theory developed in [RS17]. Namely, add universes, other notions of fibrations, and the traditional Yoneda embedding $\mathbf{y}_A : A \to (A^{\mathrm{op}} \to \mathrm{Space})$.

Besides Riehl–Shulman's work, we heavily rely on Licata–Shulman–Riley's modal framework, cf. Dan's recent talk! For related work in bicubical sets, cf. Matt's upcoming talk!

Outline

Simplicial spaces inside cubical space

The category $\mathbf{sSet}$ of *simplicial sets* is the category of presheaves on the category $\triangle$ of finite ordinals with monotone maps as morphisms.

The category $\mathbf{cSet}$ of *cubical sets* is the category of presheaves on the category of powers of the ordinal 2 with monotone maps as morphisms.

We want to define universes internally which, due to [LOPS18] becomes possible using tinyness of the *cubical* interval $\square^1$.
Simplicial sets form an essential subtopos of cubical sets.

This has been discussed by Sattler [Sat18], Kapulkin–Voevodsky [KV18], and Streicher-W [SW18].

One can show that this lifts to the level of ∞-toposes. Since this constitutes a topological modality sheafification becomes an internal operation ([RSS17]) which by the theory of compact types treated in [Rij18] can be expressed in rather elementary terms.

Universes of simplicial types

Start with a *strict* universe in cubical spaces [Shu19]. From this we derive:

- Simp: universe of simplicial types since we have a topological modality [RSS17]
- Cat: universe of (complete) Segal types due to our new notion of *cocartesian family*
- Space: universe of discrete types due to Riehl–Shulman's notion of covariant family

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Modalities: $\flat$ and $^{\mathrm{op}}$

However, the universes constructed this way are classifying only for the *cohesively discrete* (*crisp*) types.

We also want to have *opposite categories* A^{op}.

Hence, we introduce the modalities $\flat$ and $^{\mathrm{op}}$ as in the framework of Licata–Shulman–Riley to this theory.

We have a mode theory with c (cohesive/cubical) and $x : c \vdash f(x) : c$ (flat) as well as $x : c \vdash o(x) : c$ (representing the opposite cubical type/category) with equations

$$f(f(x)) = f(x), \quad o(f(x)) = f(x), \quad o(o(x)) = x, \quad f(o(x)) = f(x),$$

and

$$f(x \times y) = f(x) \times f(y), \quad o(x \times y) = o(x) \times o(y).$$

Operations on topes and shapes I

Problem: In order to get the Yoneda embedding, we need to get $\hom_A(a,b)$ for $a : A^{\mathrm{op}}$ and $b : A$ (for $A :: \mathrm{Cat}$).

Solution: Instead of ordinary $\hom$-types construct a covariant fibration $\mathrm{Tw}(A) \to A^{\mathrm{op}} \times A$ and obtain the "hom type" as the fiber. Here, $\mathrm{Tw}(A)$ is the *twisted arrow type* (traditionally, the category of elements of the uncurried Yoneda embedding) with 0-simplices

$$a \xrightarrow{f} b$$

and 1-simplices:

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle \varphi_0}\uparrow & & \downarrow{\scriptstyle \varphi_1} \\ c & \xrightarrow[g]{} & d \end{array}$$

Classically, the twisted arrow space is defined by reindexing along the functor $\varepsilon := \mathrm{op} * \mathrm{id} : \mathbb{\Delta} \to \mathbb{\Delta}$. This does not yield an extension type in Riehl–Shulman's sense.

Operations on topes and shapes II

We get the twisted arrow type using *right adjoint types* (U*-types*) in the sense of [LRS19].

First, we axiomatize operations on topes and shapes according to

$$\frac{f : \Xi \to \Xi' \quad \Xi|\Phi \vdash \Phi' f}{\Xi|\Phi \xrightarrow{f} \Xi'|\Phi'} \quad \text{and} \quad \frac{F \text{ oper} \quad \{\Xi \mid \varphi\} \text{ shape}}{F\{\Xi \mid \varphi\} \text{ shape}}.$$

Defining opposites and join for topes, we can then lift these to the level of shapes as

$$\{I \mid \varphi\} * \{J \mid \psi\} := \{I + 1 + J \mid \varphi * \psi\}, \quad \{I \mid \varphi\}^{\text{op}} = \{I \mid \varphi^{\text{op}}\}.$$

From this, we can define $\varepsilon := \text{op} * \text{id}$ for shapes. Unary operations induce modalities on the base category, hence we can define the twisted arrow types as U-types w.r.t. to ε.

Fibrational framework à Licata–Shulman–Riley

After the work of Licata–Shulman–Riley consider a type theory fibered over a type theory of modes:

Given a shape Φ and an arbitrary mode context γ, we get a universe $\gamma \vdash c_\Phi$ of small types over Φ.

For any small type $\gamma \vdash n : c_\Phi$ there is a small type $\gamma \vdash T(\Phi)(n)$ of contexts over n, a *comprehension object* in the sense of [LRS19].

Endomorphisms $f \colon \mathbb{\square} \to \mathbb{\square}$ give rise to mode morphisms $n : c_\Phi \vdash f\, n : c_{f\Phi}$.

Some rules of the type theory of modes I

$$\boxed{\gamma \vdash}$$

$$\frac{}{\cdot \vdash} \qquad \frac{\gamma \vdash \qquad \gamma \vdash n : c_\Phi}{\gamma, x : T(\Phi)(n) \vdash}$$

$$\boxed{\gamma \vdash a}$$

$$\frac{\gamma \vdash}{\gamma \vdash c_\Phi} \quad (\Phi \text{ shape}) \qquad \frac{\gamma \vdash n : c_\Phi}{\gamma \vdash T(\Phi)(n)}$$

$$\boxed{\gamma \vdash n : a}$$

$$\frac{\gamma \vdash}{\gamma \vdash \emptyset : c_\Phi} \qquad \frac{\gamma \vdash n : c_\Phi \qquad \gamma \vdash m : T(\Phi)(n)}{\gamma \vdash n.\, m : c_\Phi} \qquad \frac{\gamma \vdash n : c_\Phi}{\gamma \vdash 1 : T(\Phi)(n)} \qquad \frac{\gamma \vdash n : c_\Phi \qquad f \ \mathsf{op}_1}{\gamma \vdash f(n) : c_{f\Phi}}$$

$$\frac{\gamma \vdash \cdot \qquad f \ \mathsf{op}_0}{\gamma \vdash f_0 : c_\Phi} \qquad \frac{\gamma \vdash n : c_\Phi \qquad \gamma \vdash m : T(\Phi)(n)}{\gamma \vdash f(m) : T(f\Phi)(fn)} \qquad f(n.\, m) \equiv f(n).\, f(m)$$

Some rules of the type theory of modes II

$$\boxed{\gamma \vdash n \Rightarrow m : a}$$

$$\frac{\gamma \vdash n : c_\Phi}{\gamma \vdash 1 \Rightarrow f(1) : T(f\Phi)(fn)}$$

Some rules of the type theory–on–top

$$\boxed{\Gamma \vdash_{\gamma}}$$

$$\frac{}{\cdot \vdash_{\overline{\cdot}}} \qquad \frac{\Gamma \vdash_{\overline{\gamma \vdash T(\Phi)(n)}} A \qquad \gamma \vdash n : c_{\Phi}}{\Gamma, x : A \vdash_{\overline{\gamma, x : T(\Phi)(n)}}}$$

$$\boxed{\Gamma \vdash_{\overline{\gamma \vdash a}} A}$$

$$\frac{\Gamma \vdash_{\overline{\gamma \vdash a}} A_1 \qquad \Gamma \vdash_{\overline{\gamma \vdash a}} A_2}{\Gamma \vdash_{\overline{\gamma \vdash a}} A_1 + A_2} \qquad \frac{\Gamma \vdash_{\overline{\gamma \vdash a}} A_1 \qquad \Gamma \vdash_{\overline{\gamma \vdash a}} A_2}{\Gamma \vdash_{\overline{\gamma \vdash a}} A_1 \times A_2}$$

$$\boxed{\Gamma \vdash_{\overline{\gamma \vdash n : a}} N : A}$$

$$\frac{\Gamma \vdash_{\overline{\gamma \vdash T(\Phi)(n)}} A \qquad \gamma \vdash n : c_{\Phi}}{\Gamma, x : A \vdash_{\overline{\gamma, x : T(\Phi)(n) \vdash x : T(\Phi)(n)}} x : A}$$

Semantics of the fibrational framework I

- Mode contexts γ are (modeled as) toposes (with sufficient homotopical/logical structure).
- Modes–in–context $\gamma \vdash a$ are geometric morphisms $\mathcal{E} \to [\![\gamma \vdash]\!]$.
- Types–over–modes $\Gamma \vdash_{\overline{\gamma\vdash}}$ are objects of $[\![\gamma \vdash]\!]$.
- Terms–over–mode terms $\Gamma \vdash_{\overline{\gamma\vdash a}} A$ are objects of the fibers $\mathcal{E}_{[\![\Gamma]\!]}$.

$$\begin{array}{ccc} \mathcal{E} & & \\ {\scriptstyle [\![\gamma\vdash a]\!]}\big\downarrow \;\big\uparrow{\scriptstyle s=[\![\gamma\vdash n:a]\!]} & & \\ [\![\gamma \vdash]\!] & & \end{array} \qquad \begin{array}{c} [\![\Gamma \vdash_{\overline{\gamma\vdash a}} A]\!] \in \mathcal{E}_{[\![\Gamma]\!]} \\ \\ [\![\Gamma \vdash_{\overline{\gamma\vdash}}]\!] \in [\![\gamma \vdash]\!] \end{array} \qquad s([\![\Gamma \vdash_{\overline{\gamma\vdash}}]\!]) \xrightarrow{[\![\Gamma \vdash_{\overline{\gamma\vdash n:a}} N:A]\!]} [\![\Gamma \vdash_{\overline{\gamma\vdash a}} A]\!]$$

- The empty mode context $\cdot \vdash$ is the terminal topos.

Semantics of the fibrational framework II

- Universes $\gamma \vdash c$ are projections

$$\llbracket \gamma \rrbracket \times \mathcal{E} \xrightarrow{\llbracket \gamma \vdash c \rrbracket} \llbracket \gamma \rrbracket \qquad \llbracket \gamma \rrbracket \times \mathcal{E} \xleftarrow{\llbracket \gamma \vdash \emptyset : c \rrbracket} \llbracket \gamma \rrbracket$$

  with canonical section $\llbracket \gamma \vdash \emptyset : c \rrbracket = \lambda X.\, \langle X, 1 \rangle$.
- Comprehension objects $\gamma \vdash T(n)$ are interpreted by *Artin glueing* of $\llbracket \gamma \vdash n : a \rrbracket$:

$$\begin{array}{ccc} \llbracket \gamma, x : T(n) \vdash \rrbracket & \longrightarrow & \mathcal{E}^{\rightarrow} \\ {\scriptstyle \llbracket \gamma \vdash T(n) \rrbracket} \downarrow & \lrcorner & \downarrow {\scriptstyle \mathrm{cod}} \\ \llbracket \gamma \vdash \rrbracket & \xrightarrow[\llbracket \gamma \vdash n : a \rrbracket]{} & \mathcal{E} \end{array}$$

In particular, in our intended model of cubical spaces mode contexts will be of the form $\mathbf{cSp}/\Phi$ for a shape Φ.

Outline

Right adjoint types I

- Endomorphisms $f\colon \mathbb{\Box} \to \mathbb{\Box}$ give rise to adjoint pairs $f^*\colon \mathbf{cSp}/f\Phi \rightleftarrows \mathbf{cSp}/\Phi : f_!$, $f_! \dashv f^*$.
- The functor $f_!$ (on the level of modes) corresponds to mode morphisms.
- f^* gives rise to *right adjoint types*, *aka U-types*.
- We get a bijection

$$\left\{\Gamma \vdash_{\overline{\gamma \vdash f(k):T(f\Phi)(f\,n)}} a : A\right\} \xleftrightarrow{1:1} \left\{\Gamma \vdash_{\overline{\gamma \vdash k:T(\Phi)(n)}} b : U_f A\right\}.$$

Right adjoint types II

$$\frac{\Gamma \vdash_{\gamma \vdash T(f\Phi)(f\,n)} A \qquad \gamma \vdash n : c_\Phi}{\Gamma \vdash_{\gamma \vdash T(\Phi)(n)} U_f A} \text{ U-Form}$$

$$\frac{\Gamma \vdash_{\gamma \vdash f(k) : T(f\Phi)(f\,n)} A \qquad \gamma \vdash n : c_\Phi \qquad \gamma \vdash k : T(\Phi)(n)}{\Gamma \vdash_{\gamma \vdash k : T(\Phi)(n)} \lambda^f M : U_f A} \text{ U-Intro}$$

$$\frac{\Gamma \vdash_{\gamma \vdash k : T(\Phi)(n)} N : U_f A \qquad \gamma \vdash n : c_\Phi}{\Gamma \vdash_{\gamma \vdash f(k) : T(f\Phi)(n)} N()_f : A} \text{ U-Elim}$$

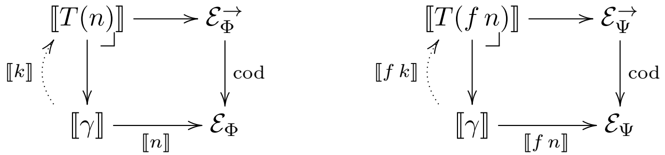
$$\lambda^f N()_f \equiv N \qquad \lambda^f M()_f \equiv M$$

Right adjoint types III

One can show that the action of the mode morphism when forming a U-type builds upon the structure of a dependent right adjoint, cf. [BCM18] *et al.*, 2018:

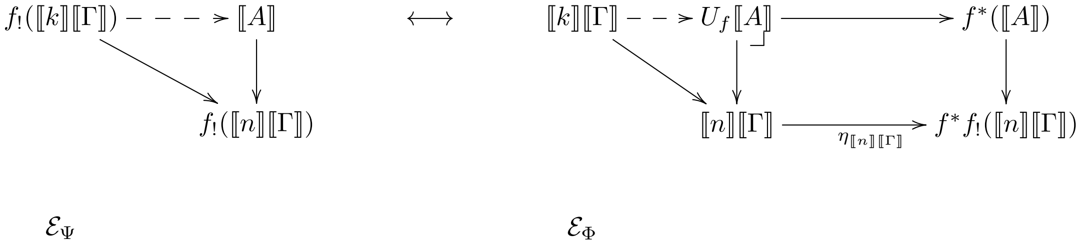
Assume an operation $f : \Phi \to \Psi$, inducing $f_! \dashv f^* : \mathcal{E}_\Phi \to \mathcal{E}_\Psi$.

For $[\![\Gamma]\!] \in [\![\gamma]\!]$ and $[\![n]\!] : [\![\gamma]\!] \to \mathcal{E}_\Phi$, consider $[\![A]\!]$ and $[\![k]\!]$ as in:



$[\![A]\!] \in \mathcal{E}_\Psi / [\![f\, n]\!][\![\Gamma]\!]$

Then we get a correspondence:



Twisted arrow types I

Externally, the twisted arrow simplicial space is constructed by reindexing along the functor $\varepsilon := \mathrm{op} * \mathrm{id}$. Thus, we internalize it by considering the U-type w.r.t. the endofunctor ε. Note that there are two natural transformations $\eta_0 : \mathrm{op} \Rightarrow \varepsilon \Leftarrow \mathrm{id} : \eta_1$ in particular, for any shape Φ giving rise to a diagram:

$$\begin{array}{ccc} \square/\Phi & \xrightarrow{\quad\varepsilon_!\quad} & \square/\varepsilon\Phi \\ & {\scriptstyle \mathrm{op}_!}\searrow \qquad \nearrow{\scriptstyle \eta_0\Phi} & \\ & \square/\,\mathrm{op}\,\Phi & \end{array}$$

Twisted arrow types II

$$\dfrac{\Gamma \vdash_{\gamma \vdash T(\varepsilon\Phi)(\varepsilon\, n)} A \qquad \Gamma \vdash_{\gamma \vdash k : T(\Phi)(n)} a_0 : (U_{\eta_0 \Phi} A)^{\mathrm{op}} \qquad \Gamma \vdash_{\gamma \vdash k : T(\Phi)(n)} a_1 : U_{\eta_1 \Phi} A \qquad \gamma \vdash k : T(\Phi)(n) \qquad \gamma \vdash n : c_\Phi}{\Gamma \vdash_{\gamma \vdash T(\Phi)(n)} \mathrm{tw}^k_A(a_0, a_1)} \text{ tw-Form}$$

$$\dfrac{\Gamma \vdash_{\gamma \vdash T(\varepsilon\Phi)(\varepsilon\, n)} A \qquad \Gamma \vdash_{\gamma \vdash \varepsilon(k) : T(\varepsilon\Phi)(\varepsilon\, n)} a : A \qquad \left(\lambda^{\eta_0 \Phi} a\right)^{\mathrm{op}} \equiv a_0 \qquad \lambda^{\eta_1 \Phi} a \equiv a_1 \qquad \gamma \vdash k : T(\Phi)(n) \qquad \gamma \vdash n : c_\Phi}{\Gamma \vdash_{\gamma \vdash k : T(\Phi)(n)} \lambda^{\mathrm{tw}} a : \mathrm{tw}^k_A(a_0, a_1)} \text{ tw-Intro}$$

$$\dfrac{\Gamma \vdash_{\gamma \vdash k : T(\Phi)(n)} b : \mathrm{tw}^k_A(a_0, a_1) \qquad \gamma \vdash k : T(\Phi)(n) \qquad \gamma \vdash n : c_\Phi}{\Gamma \vdash_{\gamma \vdash \varepsilon(k) : T(\varepsilon\Phi)(\varepsilon\, n)} b(\,)_{\mathrm{tw}} : A} \text{ tw-Elim}$$

$$\left(\lambda^{\eta_0\ \Phi} b(\,)_{\mathrm{tw}}\right)^{\mathrm{op}} \equiv a_0 \qquad \lambda^{\eta_1\ \Phi} b(\,)_{\mathrm{tw}} \equiv a_1 \qquad \lambda^{\mathrm{tw}} a(\,)_{\mathrm{tw}} \equiv a \qquad \lambda^{\mathrm{tw}} b(\,)_{\mathrm{tw}} \equiv b$$

Twisted arrow types III

Using that the flat modality can be defined as the U-type w.r.t. the terminal projection functor $! \colon ⎕ \to ⎕$ one can show for crisp Segal types A that e.g. $\flat \hom_A(a_0, a_1) \simeq \flat \mathrm{tw}_A(a_0, a_1)$ using the ensuing computation rules for U-types.

Outline

Perspectives

Work in progress:

- Give a full proof of an analog of the “classical Yoneda Lemma” using twisted arrow types.
- Define fibrancy structures internally on the universes Simp, Cat, and Space, possibly á la Orton (PhD thesis).
- Do we get (enough of) the expected 2-dimensional structure for the theory of Segal/Rezk types, cf. Riehl–Shulman, Riehl–Verity's ∞-cosmos theory?

Based on the same frameworks:

- Cavallo–Riehl–Sattler: Directed univalence for simplicial type theory [CRS18]
- Licata–Weaver: Directed univalence for bicubical directed type theory [LW18]

Selection of further work on directed type theory:

- Altenkirch–Sestini: “Naturality for free", 2019)
- Cavallo–Harper: parametric CTT, 2019)
- North: directed HoTT & wfs, 2018/19
- Nuyts: directed HoTT, 2015+; w/ Devriese: Menkar, ultimode presheaf proof assistant `https://github.com/anuyts/menkar`

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Thank you!