Calculational HoTT
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Content

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The result: "Calculational HoTT"(arXiv:1901.08883v2), a joint work with Bernarda Aldana and Jaime Bohorquez.

Equational axioms and Leibniz rules

Brief description of CL.

Main feature:

CL is an equational logical system

CL axioms are logical equations

$$A \equiv B, \; C \equiv D, \ldots$$

CL inference rules are Leibniz's rules

$$\frac{E[x/A] \quad A \equiv B}{E[x/B]}$$

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Calculations

Derivations in CL are deduction trees of the form:

$$\cfrac{\cfrac{\cfrac{E_1 \qquad A \equiv B}{E_2} \qquad C \equiv D}{E_3} \qquad E \equiv F}{E_4}$$

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This deduction tree, written vertically, is what Lifschitz called 'Calculation'[Lifs]:

$$\begin{array}{ll} & E_1 \\ \Leftrightarrow & \quad \langle\, A \equiv B \,\rangle \\ & E_2 \\ \Leftrightarrow & \quad \langle\, C \equiv D \,\rangle \\ & E_3 \\ \Leftrightarrow & \quad \langle\, E \equiv F \,\rangle \\ & E_4 \end{array}$$

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There are sound and complete calculational versions of both, classical (CCL) and intuitionistic (ICL) first order logic.

Embeddings

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- establishing a linear calculation format as an instrument to understand proofs in HoTT book, and
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Note: We expected to be more comfortable with a linear calculation format as an instrument to understand proofs in HoTT book.

Deductive chains

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$A_1 \simeq A_3 <:$	if none of the premises is of the form $A \to B <:$ and at least one is of the form $A \simeq B <:$
$A_1 \equiv A_3$	if all the premises are of the form $A \equiv B$

Deductive chains

By induction we have the following derivation

$$\dfrac{\begin{array}{c}\vdots\\ a : A_1\end{array} \qquad \begin{array}{c}\vdots\\ A_1 \rightsquigarrow A_2\end{array} \qquad \cdots \qquad \begin{array}{c}\vdots\\ A_{n-1} \rightsquigarrow A_n\end{array}}{A_n <:} \; .$$

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which may be represented vertically by the following format-scheme

$$\begin{array}{ll} & A_n \\ \leftrightarrows & \quad \langle \cdots \rangle \\ & A_{n-1} \\ \vdots & \\ & A_2 \\ \leftrightarrows & \quad \langle \cdots \rangle \\ & A_1 \\ \hat{:} & \quad \langle \cdots \rangle \\ & a \end{array}$$

which we called a *deductive chain*.

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The link at the bottom of the deductive chain is called *inhabitation link*.

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Unified notation for operationals

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-Summation:

$$(\Sigma i : \mathbb{N} \mid 1 \leq i \leq 3 \cdot i^2) = 1^2 + 2^2 + 3^2$$

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[**Trade**] rules

$$(\forall x : T \,|\, P \cdot Q) \equiv (\forall x : T \cdot P \Rightarrow Q)$$

$$(\exists x : T \,|\, P \cdot Q) \equiv (\exists x : T \cdot P \wedge Q)$$

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[One-Point]:
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[One-Point]:

$$\boxed{\begin{array}{l}\prod_{x:A} \prod_{p:x=a} P(x,p) \simeq P(a, \mathrm{refl}_a) <: \\ \\ \sum_{x:A} \sum_{p:x=a} P(x,p) \simeq P(a, \mathrm{refl}_a) <:\end{array}} \quad \text{(HoTT)}$$

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[Equality]:

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[Range Split]:

$$\boxed{\begin{array}{c}(\forall x : T \mid P \vee Q \cdot R) \equiv (\forall x : T \mid P \cdot R) \wedge (\forall x : T \mid Q \cdot R) \\ \\ (\exists x : T \mid P \vee Q \cdot R) \equiv (\exists x : T \mid P \cdot R) \vee (\exists x : T \mid Q \cdot R)\end{array}}$$

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$$\boxed{\begin{array}{c}\prod_{x:A+B} P(x) \simeq \prod_{x:A} P(\mathrm{inl}(x)) \times \prod_{x:B} P(\mathrm{inr}(x)) <: \\ \\ \sum_{x:A+B} P(x) \simeq \sum_{x:A} P(\mathrm{inl}(x)) + \sum_{x:B} P(\mathrm{inr}(x)) <:\end{array}}$$

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[Term Split]:

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$$\boxed{\begin{array}{l}\prod_{x:A}(P(x) \times Q(x)) \simeq \prod_{x:A} P(x) \times \prod_{x:A} Q(x) <: \\[1em] \sum_{x:A}(P(x) + Q(x)) \simeq \sum_{x:A} P(x) + \sum_{x:A} Q(x) <:\end{array}}$$

ICL quatified axioms and theorems

[Translation]
$$\boxed{\begin{array}{c}(\forall x\!:\!J \mid P \cdot Q) \equiv (\forall y\!:\!K \mid P[f(y)/x] \cdot Q[f(y)/x]) \\ \\ (\exists x\!:\!J \mid P \cdot Q) \equiv (\exists y\!:\!K \mid P[f(y)/x] \cdot Q[f(y)/x]),\end{array}}$$

where f is a bijection that maps values of type K to values of type J.

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[Antecedent] $\boxed{\begin{array}{c}R \Rightarrow (\forall x:T \mid P \cdot Q) \equiv (\forall x:T \mid P \cdot R \Rightarrow Q) \\ \\ R \Rightarrow (\exists x:T \mid P \cdot Q) \equiv (\exists x:T \mid P \cdot R \Rightarrow Q)\end{array}}$

when there are not free occurrences of x in R.

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[Leibniz principles] $$\boxed{\begin{array}{c}(\forall x, y\!:\!T \mid x = y \cdot f(x) = f(y)) \\ \\ (\exists x, y\!:\!T \mid x = y \cdot P(x) \equiv P(y))\end{array}}$$
where f is a function that maps values of type T to values of any other type and P is a predicate.

Equational judgments in HoTT

[Translation] $\boxed{\begin{array}{c}\prod_{x:A} P(x) \simeq \prod_{y:B} P(g(y)) <: \\ \\ \sum_{x:A} P(x) \simeq \sum_{y:B} P(g(y)) <:\end{array}}$

where g is an inhabitant of $B \simeq A$.

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when R does not depend on x.

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$$\boxed{\begin{array}{c}\prod_{x,y:A} x = y \to f(x) = f(y) <: \\ \\ \prod_{x,y:A} x = y \to P(x) \simeq P(y) <:\end{array}}$$
where $f : A \to B$ and $P : A \to \mathcal{U}$ is a type family.

A deduction

I will derive the judgment

$$(\prod_{x:A} \prod_{y:B(x)} P((x,y))) \simeq \prod_{g:\sum_{x:A} B(x)} P(g) <: \tag{1}$$

which corresponds to the homotopic equivalence version of the Σ induction operator.

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which corresponds to the homotopic equivalence version of the Σ induction operator.

Note. The ICL theorem corresponding to (1), when P is a non-dependent type, is

$$(\forall x:T \mid B \cdot P) \equiv (\exists x:T \cdot B) \Rightarrow P$$

where x does not occur free in P.

This motivate us to call the equivalence Σ-[**Consequent**] rule.

A deduction

Recall that the Σ-induction operator

$$\boldsymbol{\sigma} : \Big(\prod_{x:A}\prod_{y:B(x)} P((x,y))\Big) \to \prod_{g:\sum_{x:A} B(x)} P(g)$$

is defined by

$$\boldsymbol{\sigma}(u)((x,y)) :\equiv u(x)(y).$$

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Let

$$\Phi : (\prod_{g:\sum_{x:A} B(x)} P(g)) \to \prod_{x:A} \prod_{y:B(x)} P((x,y))$$

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Then $\Phi \circ \boldsymbol{\sigma}$ is homotopic to the identity function of $\prod_{x:A}\prod_{y:B(x)} P((x,y))$.

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So, $\boldsymbol{\sigma} \circ \Phi$ is homotopic to the identity function of $\prod_{g:\sum_{x:A} B(x)} P(g)$.

This proves the Σ-[**Consequent**] rule.

Example

Application of Π-translation rule (to prove isSet($\mathbb{N}$) <:). We can use the translation rule to prove isSet($\mathbb{N}$) <:
In fact, let $\Phi : m = n \to \mathrm{code}(m,n)$ be defined by $\Phi :\equiv \mathrm{encode}(m,n)$ and let $\Psi : \mathrm{code}(m,n) \to m = n$ be defined by $\Psi :\equiv \mathrm{decode}(m,n)$. Then,

$$
\begin{array}{ll}
 & \mathrm{isSet}(\mathbb{N}) \\
\equiv & \quad \langle \text{ Definition of isSet } \rangle \\
 & \displaystyle\prod_{m,n:\mathbb{N}} \prod_{p,q:m=n} p = q \\
\simeq & \quad \langle \text{ Π-translation rule} \,;\, m = n \simeq \mathrm{code}(m,n) \ \rangle \\
 & \displaystyle\prod_{m,n:\mathbb{N}} \prod_{s,t:\mathrm{code}(m,n)} \Psi(s) = \Psi(t) \\
\overset{\wedge}{\vdots} & \quad \langle \text{ See definition of } h \text{ below } \rangle \\
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\end{array}
$$

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1. Deductive chains are really formal linear tools to prove theorems in HoTT.
2. There is an embedding of ICL in HOTT. In particular we found that the Eindhoven quantifiers correspond to the main dependent types in HoTT.
3. We found strong evidence that it is possible to restate the whole of HoTT giving equality and homotopic equivalence a preeminent role, both, axiomatically and proof-theoretically.

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