

# Quillen model structures on cubical sets

Steve Awodey

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# Acknowledgements

- ▶ Parts are joint work with Coquand and Riehl.
- ▶ Parts are also joint with Cavallo and Sattler.
- ▶ Ideas are also borrowed from Joyal and Orton-Pitts.

# Models of HoTT from QMS

The first models of HoTT were built from Quillen model categories.

- ▶ A-Warren: general Quillen model structures and weak factorization systems
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In each case, more specific QMS led to “better” models of type theory, with coherent  $\text{Id}$ ,  $\Sigma$ ,  $\Pi$  and eventually univalent  $U$ .

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## Definition (*pace Orton-Pitts*)

A *premodel of HoTT* consists of  $(\mathcal{E}, \Phi, \mathbb{I}, \mathbb{V})$  where:

- ▶  $\mathcal{E}$  is a topos
- ▶  $\Phi$  is a representable class of monos  $\Phi \multimap \Omega$  that form a *dominance* and ...
- ▶  $\mathbb{I}$  is an interval  $1 \rightrightarrows \mathbb{I}$  in  $\mathcal{E}$  that is *tiny*  $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$  and ...
- ▶  $\dot{\mathbb{V}} \rightarrow \mathbb{V}$  is a universe of *small families*, closed under  $\Sigma, \Pi$  and ...

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A model of HoTT is then constructed internally using the *extensional* type theory of  $\mathcal{E}$  (see Orton-Pitts).

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Our goal here is to show that from such a set-up for modelling HoTT one can also construct a QMS:

## Construction

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The resulting QMS is right proper and has descent, so it also admits a model of HoTT in the pre-Orton-Pitts sense.

# QMS from models of HoTT

The construction of a QMS  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  from a premodel  $(\mathcal{E}, \Phi, \mathbb{I}, \mathbf{V})$  is general, but the details depend on the specifics of the premodel.

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We consider three special cases of *cubical sets*.

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1. Cartesian cubical sets (new)
2. Cartesian cubical sets with equivariance (new jww/CCRS)
3. Dedekind cubical sets (Sattler)

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1. use  $\Phi$  to determine an awfs  $(\mathcal{C}, \mathbf{TFib})$ ,
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To prove the Fibration Extension Property:

4. show that  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  satisfies the EEP,
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**NB:** (5) seems to be a detour; maybe one can prove FEP directly?

# 1. The cofibration awfs $(\mathcal{C}, \text{TFib})$

The monos classified by  $\Phi \twoheadrightarrow \Omega$  are called *cofibrations*.

The generic one  $1 \twoheadrightarrow \Phi$  determines a polynomial endofunctor,

$$X^+ := \sum_{\varphi:\Phi} X^\varphi,$$

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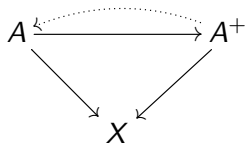
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Algebras for the pointed endofunctor of this monad,



form the right class of an awfs – they are the *trivial fibrations*.

## 2. The fibration awfs $(\text{TCof}, \mathcal{F})$

For any  $c : A \rightarrow B$  in  $\text{cSet}^2$ , the *Leibniz adjunction*

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### Definition

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This notion of fibration is used for the *Dedekind cubes*.



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For the *Cartesian cubes*, we pass to the slice category  $\mathrm{cSet}/\mathbb{I}$ , where there is a *generic point*  $\delta : 1 \rightarrow \mathbb{I}$ .

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### Remark

There is also an *equivariant* version of this awfs, in which the fibration structure respects the symmetries of the cubes  $\mathbb{I}^n$  (this is explained in Emily's talk).

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We will compare  $\mathcal{W}$  with the following, which does satisfy 3-for-2.

#### Definition

A map  $f : Y \rightarrow X$  is a *weak homotopy equivalence* if the map

$$K^f : K^X \longrightarrow K^Y$$

is a bijection on connected components for all fibrant objects  $K$ .

# The QMS $(\mathcal{C}, \mathcal{W}, \mathcal{F})$

## Definition (FEP)

The *Fibration Extension Property* says that fibrations extend along trivial cofibrations:

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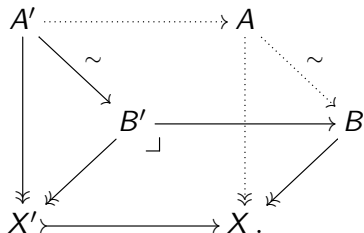
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## 4. The equivalence extension property

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The diagram shows a commutative square with two triangles. At the top left is  $A'$  and at the top right is  $A$ . A dotted arrow points from  $A'$  to  $A$ . Below  $A'$  is  $X'$  and below  $A$  is  $X$ . A solid arrow points from  $X'$  to  $X$ . On the left, a solid arrow points from  $A'$  to  $X'$ . On the right, a solid arrow points from  $A$  to  $X$ . In the middle, there are two triangles. The left triangle has vertices  $A'$ ,  $B'$ , and  $X'$ . A solid arrow points from  $A'$  to  $B'$ , and another solid arrow points from  $B'$  to  $X'$ . A tilde symbol  $\sim$  is placed above the arrow from  $A'$  to  $B'$ . The right triangle has vertices  $A$ ,  $B$ , and  $X$ . A dotted arrow points from  $A$  to  $B$ , and another dotted arrow points from  $B$  to  $X$ . A tilde symbol  $\sim$  is placed above the arrow from  $A$  to  $B$ . A solid arrow points from  $B'$  to  $B$ . A right-angle symbol  $\perp$  is placed below the arrow from  $B'$  to  $B$ .

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This was shown by Voevodsky for modelling univalence in Kan simplicial sets. A related proof by Sattler works in our setting.

## 5. The universe $U$ of fibrations

There is a *universal (small) fibration*  $\dot{U} \twoheadrightarrow U$ .

Every small fibration  $A \twoheadrightarrow X$  is a pullback of  $\dot{U} \twoheadrightarrow U$  along a canonical classifying map  $X \rightarrow U$ .

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Then define  $\dot{U} \rightarrow U$  by pulling back the universal small family.

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But  $\text{Fib}(-)$  is stable under pullback, so there is a section

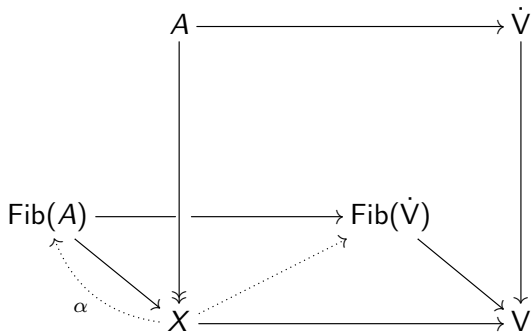
$$\begin{array}{ccc} \dot{U} & \longrightarrow & \dot{V} \\ \downarrow \lrcorner & & \downarrow \\ U & \longrightarrow & V \\ \uparrow & & \uparrow \\ \text{Fib}(\dot{U}) & \longrightarrow & \text{Fib}(\dot{V}) \end{array}$$

(A dashed arrow points from  $\text{Fib}(\dot{U})$  to  $U$ .)

Thus  $\dot{U} \rightarrow U$  is a fibration.

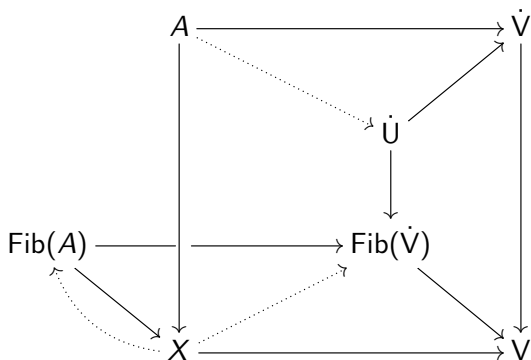
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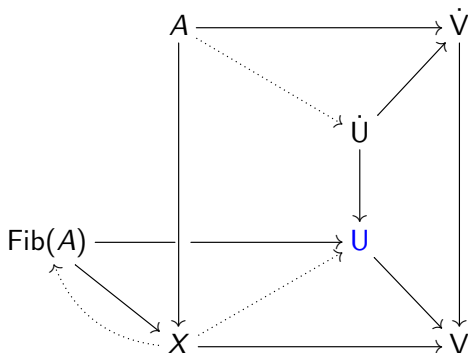
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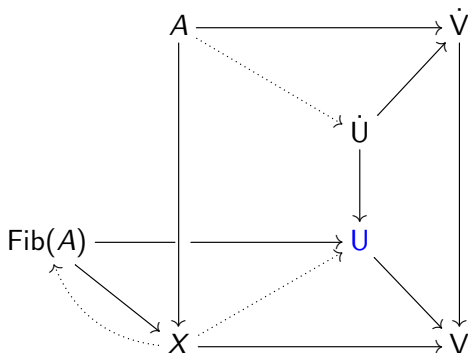
## 5. The universe $U$ of fibrations

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The construction of  $Fib$  uses the *root functor*  $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$ .

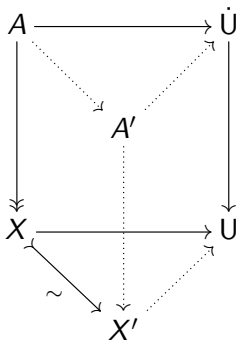
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The FEP says just that  $U$  is fibrant:

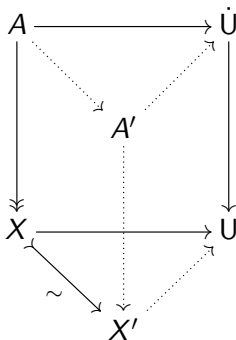




## FEP and EEP in terms of U

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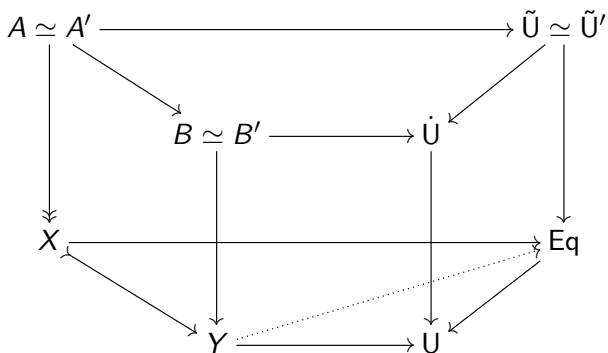
The FEP says just that  $U$  is fibrant:



Voevodsky proved this for Kan simplicial sets.

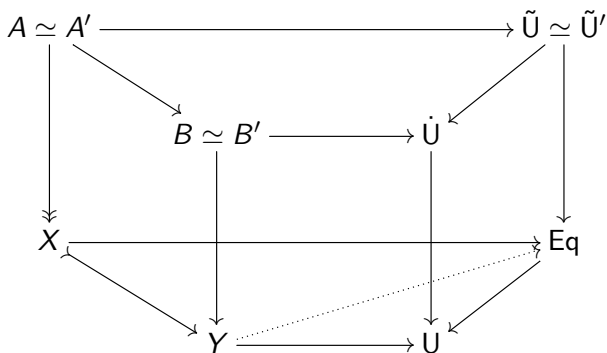
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The EEP says that  $\text{Eq} \longrightarrow \text{U}$  is a TFib:



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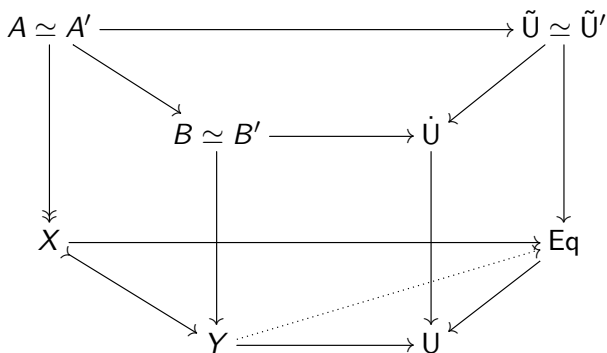
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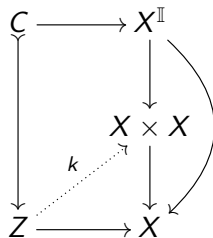
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## 6. FEP from EEP

Coquand gave a proof of FEP from EEP using *Kan composition*.

### Definition

An object  $X$  has (*biased*) *composition* if for every cofibration  $C \twoheadrightarrow Z$  and commutative rectangle as on the outside below,



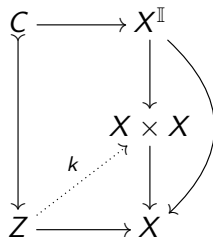
there is an arrow  $k : Z \longrightarrow X \times X$  making the diagram commute.

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there is an arrow  $k : Z \rightarrow X \times X$  making the diagram commute.

### Lemma

If  $X$  has composition, then  $X$  is fibrant. □

## 6. FEP from EEP

We can now show:

### Proposition

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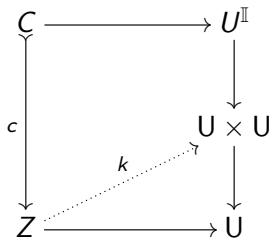
By the previous lemma it suffices to show:

### Lemma

*The universe  $U$  has composition.*

### Proof.

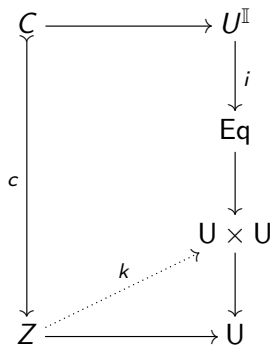
Consider a composition problem





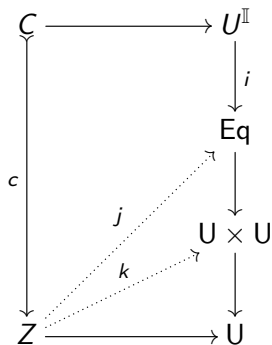
## 6. FEP from EEP

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$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad} & U^{\mathbb{I}} \\ \downarrow c & & \downarrow i \\ & & \text{Eq} \\ & \nearrow j & \downarrow \\ & & U \times U \\ & \nearrow k & \downarrow \\ Z & \xrightarrow{\quad} & U \end{array}$$

But the projection  $\text{Eq} \rightarrow U$  is a trivial fibration by EEP, so there is a diagonal filler  $j$ . Composing gives the required  $k$ .  $\square$

Done!

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But is our QMS *right proper*?

# Postscript: Frobenius

## Definition (Frobenius)

The *Frobenius Property* says that trivial cofibrations pull back along fibrations,

$$\begin{array}{ccc} A' & \longrightarrow & X' \\ \downarrow \sim & \lrcorner & \downarrow \sim \\ A & \twoheadrightarrow & X \end{array}$$

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It is equivalent to the condition that fibrations “push forward” along fibrations,

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This is related to the existence of  $\Pi$ -types. It implies that our QMS is *right proper*.



# Frobenius

## Proposition

*The Frobenius property holds for  $(\text{TCof}, \mathcal{F})$ .*

Proof.

$$\begin{array}{ccccccc} B^{\mathbb{I}} & \xrightarrow{\delta \Rightarrow B} & B_{\epsilon}^* & \longrightarrow & B_{\epsilon} & \longrightarrow & B \\ & \searrow & \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \\ & & A^{\mathbb{I}} & \xrightarrow{\delta \Rightarrow A} & A_{\epsilon} & \longrightarrow & A \\ & & & \searrow & \downarrow \lrcorner & & \downarrow \\ & & & & X^{\mathbb{I}} & \xrightarrow{\epsilon} & X \end{array}$$

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 & & & \searrow & \downarrow \lrcorner & & \downarrow \\
 & & & & X^{\mathbb{I}} & \xrightarrow{\epsilon} & X \\
 & & & \nearrow & \uparrow & & \uparrow \\
 (\Pi_A B)^{\mathbb{I}} & \xrightarrow{\delta \Rightarrow \Pi_A B} & (\Pi_A B)_{\epsilon} & \longrightarrow & \Pi_A B & & \\
 \uparrow & & \uparrow & & \uparrow & & \\
 \Pi_{A^{\mathbb{I}}} B^{\mathbb{I}} & \longrightarrow & \Pi_{A^{\mathbb{I}}} B_{\epsilon}^* & & & & 
 \end{array}$$

The diagram illustrates the Frobenius property proof. It consists of several commutative diagrams and maps:
 

- Top row:  $B^{\mathbb{I}} \xrightarrow{\delta \Rightarrow B} B_{\epsilon}^* \longrightarrow B_{\epsilon} \longrightarrow B$
- Second row:  $A^{\mathbb{I}} \xrightarrow{\delta \Rightarrow A} A_{\epsilon} \longrightarrow A$
- Third row:  $X^{\mathbb{I}} \xrightarrow{\epsilon} X$
- Bottom row:  $(\Pi_A B)^{\mathbb{I}} \xrightarrow{\delta \Rightarrow \Pi_A B} (\Pi_A B)_{\epsilon} \longrightarrow \Pi_A B$
- Bottom-most row:  $\Pi_{A^{\mathbb{I}}} B^{\mathbb{I}} \longrightarrow \Pi_{A^{\mathbb{I}}} B_{\epsilon}^*$
- Vertical maps:  $B^{\mathbb{I}} \rightarrow A^{\mathbb{I}} \rightarrow X^{\mathbb{I}}$ ;  $B_{\epsilon}^* \rightarrow A^{\mathbb{I}}$ ;  $B_{\epsilon} \rightarrow A_{\epsilon}$ ;  $A_{\epsilon} \rightarrow X^{\mathbb{I}}$ ;  $(\Pi_A B)^{\mathbb{I}} \rightarrow X^{\mathbb{I}}$ ;  $(\Pi_A B)_{\epsilon} \rightarrow X^{\mathbb{I}}$ ;  $\Pi_{A^{\mathbb{I}}} B^{\mathbb{I}} \rightarrow (\Pi_A B)^{\mathbb{I}}$ ;  $\Pi_{A^{\mathbb{I}}} B_{\epsilon}^* \rightarrow (\Pi_A B)_{\epsilon}$ .
- Right-side vertical maps:  $B \rightarrow A \rightarrow X$ ;  $A \rightarrow X$ ;  $\Pi_A B \rightarrow X$ .
- Commutativity:  $\lrcorner$  symbols indicate commutative squares between  $(B^{\mathbb{I}}, B_{\epsilon}^*, A^{\mathbb{I}}, A)$ ,  $(B_{\epsilon}^*, B_{\epsilon}, A^{\mathbb{I}}, A)$ ,  $(B_{\epsilon}, B, A_{\epsilon}, A)$ ,  $(A^{\mathbb{I}}, A_{\epsilon}, X^{\mathbb{I}}, A)$ , and  $(A_{\epsilon}, X^{\mathbb{I}}, X, A)$ .
- Dotted arrows:  $\Pi_{A^{\mathbb{I}}} B^{\mathbb{I}} \rightarrow (\Pi_A B)^{\mathbb{I}}$  and  $\Pi_{A^{\mathbb{I}}} B_{\epsilon}^* \rightarrow (\Pi_A B)_{\epsilon}$ .



That's all Folks!