Quillen model structures on cubical sets

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Acknowledgements

- Parts are joint work with Coquand and Riehl.
- Parts are also joint with Cavallo and Sattler.
- Ideas are also borrowed from Joyal and Orton-Pitts.

Models of HoTT from QMS

The first models of HoTT were built from Quillen model categories.

- A-Warren: general Quillen model structures and weak factorization systems
- van den Berg-Garner: special weak factorization systems on spaces and simplicial sets
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In each case, more specific QMS led to "better" models of type theory, with coherent Id, Σ, Π and eventually univalent U.

QMS from models of HoTT

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Definition (*pace* Orton-Pitts)

A *premodel of HoTT* consists of $(\mathcal{E}, \Phi, \mathbb{I}, \mathsf{V})$ where:

- $\mathcal{E}$ is a topos
- Φ is a representable class of monos $\Phi \rightarrowtail \Omega$ that form a *dominance* and ...
- $\mathbb{I}$ is an interval $1 \rightrightarrows \mathbb{I}$ in $\mathcal{E}$ that is *tiny* $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$ and ...
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A model of HoTT is then constructed internally using the *extensional* type theory of $\mathcal{E}$ (see Orton-Pitts).

QMS from models of HoTT

Our goal here is to show that from such a set-up for modelling HoTT one can also construct a QMS:

Construction

From a premodel $(\mathcal{E}, \Phi, \mathbb{I}, \mathsf{V})$ *one can construct a QMS on* $\mathcal{E}$*.*

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Construction

From a premodel $(\mathcal{E}, \Phi, \mathbb{I}, \mathrm{V})$ *one can construct a QMS on* $\mathcal{E}$*.*

The resulting QMS is right proper and has descent, so it also admits a model of HoTT in the pre-Orton-Pitts sense.

QMS from models of HoTT

The construction of a QMS $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ from a premodel $(\mathcal{E}, \Phi, \mathbb{I}, \mathsf{V})$ is general, but the details depend on the specifics of the premodel.

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We consider three special cases of *cubical sets*.

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1. Cartesian cubical sets
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$$\mathcal{E} = \mathsf{Set}^{\mathbb{C}^{op}}$$

1. Cartesian cubical sets (new)
2. Cartesian cubical sets with equivariance (new jww/CCRS)
3. Dedekind cubical sets (Sattler)

Outline of the construction

Let $(\mathcal{E}, \Phi, \mathbb{I}, \mathsf{V})$ be a premodel of HoTT where $\mathcal{E} = \mathsf{cSet}$.

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We construct a Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathcal{E}$ in 3 steps:

1. use Φ to determine an awfs $(\mathcal{C}, \mathsf{TFib})$,
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To prove the Fibration Extension Property:

4. show that $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ satisfies the EEP,
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NB: (5) seems to be a detour; maybe one can prove FEP directly?

1. The cofibration awfs ($\mathcal{C}$, TFib)

The monos classified by $\Phi \rightarrowtail \Omega$ are called *cofibrations*.
The generic one $1 \rightarrowtail \Phi$ determines a polynomial endofunctor,

$$X^+ := \sum_{\varphi:\Phi} X^\varphi \,,$$

which is a (fibered) monad,

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Algebras for the pointed endofunctor of this monad,

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & A^+ \arrow[ll, bend right, dotted] \arrow[dl] \\ & X & \end{tikzcd}$$

form the right class of an awfs – they are the *trivial fibrations*.

2. The fibration awfs $(\mathsf{TCof}, \mathcal{F})$

For any $c : A \to B$ in $\mathrm{cSet}^{\mathbb{2}}$, the *Leibniz adjunction*

$$(-)\hat{\otimes} c \;\dashv\; c \hat{\Rightarrow} (-)$$

relates the pushout-product with c and the pullback-hom with c.

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Definition

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This notion of fibration is used for the *Dedekind cubes*.

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For the *Cartesian cubes*, we pass to the slice category $\mathsf{cSet}/\mathbb{I}$, where there is a *generic point* $\delta : 1 \to \mathbb{I}$.

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Proposition

There is an awfs $(\mathsf{TCof}, \mathcal{F})$ *with these fibrations as* $\mathcal{F}$*.*

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Remark

There is also an *equivariant* version of this awfs, in which the fibration structure respects the symmetries of the cubes $\mathbb{I}^n$ (this is explained in Emily's talk).

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Now define

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It is easy to show that

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so we just need the 3-for-2 property for $\mathcal{W}$.

We will compare $\mathcal{W}$ with the following, which does satisfy 3-for-2.

Definition

A map $f : Y \to X$ is a *weak homotopy equivalence* if the map

$$K^f : K^X \longrightarrow K^Y$$

is a bijection on connected components for all fibrant objects K.

The QMS $(\mathcal{C}, \mathcal{W}, \mathcal{F})$

Definition (FEP)

The *Fibration Extension Property* says that fibrations extend along trivial cofibrations:

$$\begin{tikzcd}
A \arrow[r, dotted] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, dotted, two heads] \\
X \arrow[r, tail, "\sim"'] & X'
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Lemma

If the FEP holds, then a map $f : Y \to X$ is a weak equivalence iff it is a weak homotopy equivalence.

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Corollary

If the FEP holds, then $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a QMS.

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4. The equivalence extension property

Definition (EEP)

The EEP says that weak equivalences extend along any cofibration $X' \rightarrowtail X$: given a fibration $B \twoheadrightarrow X$, and a weak equivalence $A' \simeq B'$ over X', where $A' \twoheadrightarrow X'$ and $B' = X' \times_X B$,

$$\begin{tikzcd}
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& B' \arrow[rr] \arrow[dl, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dl, two heads] \\
X' \arrow[rr, tail] & & X\,. &
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there is a fibration $A \twoheadrightarrow X$, and a weak equivalence $A \simeq B$ over X that pulls back to $A' \simeq B'$.

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This is was shown by Voevodsky for modelling univalence in Kan simplicial sets. A related proof by Sattler works in our setting.

5. The universe U of fibrations

There is a *universal (small) fibration* $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$.
Every small fibration $A \twoheadrightarrow X$ is a pullback of $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ along a canonical classifying map $X \to \mathsf{U}$.

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d, two heads] \\ X \arrow[r] & \mathsf{U} \end{tikzcd}$$

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Then define $\dot{\mathsf{U}} \to \mathsf{U}$ by pulling back the universal small family.

$$\begin{tikzcd} \dot{\mathsf{U}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{V}} \arrow[d] \\ \mathsf{U} \arrow[r] & \mathsf{V} \end{tikzcd}$$

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\end{tikzcd}
$$

But $\mathsf{Fib}(-)$ is stable under pullback, so there is a section

$$
\begin{tikzcd}
\dot{\mathsf{U}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{V}} \arrow[d] \\
\mathsf{U} \arrow[r] \arrow[d, bend right, dotted] & \mathsf{V} \\
\mathsf{Fib}(\dot{\mathsf{U}}) \arrow[r] \arrow[u] & \mathsf{Fib}(\dot{\mathsf{V}}) \arrow[u]
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Thus $\dot{\mathsf{U}} \longrightarrow \mathsf{U}$ is a fibration.

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A fibration structure α on a family $A \to X$ therefore gives rise to a factorization of the classifying map to $\dot{\mathsf{V}} \to \mathsf{V}$.

$$
\begin{tikzcd}
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A fibration structure α on a family $A \to X$ therefore gives rise to a factorization of the classifying map to $\dot{\mathsf{V}} \to \mathsf{V}$ through the fibration classifier $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$.

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 & X \arrow[rr] \arrow[ur, dotted] \arrow[ul, dotted, bend left] & & \mathsf{V}
\end{tikzcd}
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The construction of Fib uses the *root functor* $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$.

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Voevodsky proved this for Kan simplicial sets.

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The EEP says that $\mathsf{Eq} \longrightarrow \mathsf{U}$ is a TFib:

$$\begin{tikzcd}
A \simeq A' \arrow[rrr] \arrow[dr] \arrow[dd, two heads] & & & \tilde{\mathsf{U}} \simeq \tilde{\mathsf{U}}' \arrow[dl] \arrow[dd] \\
& B \simeq B' \arrow[r] \arrow[dd] & \dot{\mathsf{U}} \arrow[dd] & \\
X \arrow[rrr, hook] \arrow[dr, hook] & & & \mathsf{Eq} \arrow[dl] \\
& Y \arrow[r] \arrow[urr, dotted] & \mathsf{U} &
\end{tikzcd}$$

FEP and EEP in terms of U

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Shulman gave a neat proof of FEP from EEP, but it uses 3-for-2.

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X \arrow[rrr] \arrow[dr, tail] & & & \mathrm{Eq} \arrow[dl] \\
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\end{tikzcd}$$

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6. FEP from EEP

Coquand gave a proof of FEP from EEP using *Kan composition*.

Definition

An object X has *(biased) composition* if for every cofibration $C \rightarrowtail Z$ and commutative rectangle as on the outside below,

$$\begin{tikzcd}
C \arrow[r] \arrow[dd, tail] & X^{\mathbb{I}} \arrow[d] \arrow[dd, bend left] \\
& X \times X \arrow[d] \\
Z \arrow[r] \arrow[ur, dotted, "k"] & X
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there is an arrow $k : Z \longrightarrow X \times X$ making the diagram commute.

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$$\begin{tikzcd}
C \arrow[r] \arrow[dd, tail] & X^{\mathbb{I}} \arrow[d] \arrow[dd, bend left=60] \\
 & X \times X \arrow[d] \\
Z \arrow[r] \arrow[ur, dotted, "k"] & X
\end{tikzcd}$$

there is an arrow $k : Z \longrightarrow X \times X$ making the diagram commute.

Lemma

If X has composition, then X is fibrant. $\square$

6. FEP from EEP

We can now show:

Proposition

The universe U *is fibrant.*

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Proposition

The universe U *is fibrant.*

By the previous lemma it suffices to show:

Lemma

The universe U *has composition.*

Proof.

Consider a composition problem

$$\begin{tikzcd}
C \arrow[r] \arrow[dd, tail, "c"'] & U^{\mathbb{I}} \arrow[d] \\
 & \mathrm{U} \times \mathrm{U} \arrow[d] \\
Z \arrow[r] \arrow[ur, dotted, "k"] & \mathrm{U}
\end{tikzcd}$$

6. FEP from EEP

The canonical map $\mathsf{U}^{\mathbb{I}} \longrightarrow \mathsf{U} \times \mathsf{U}$ factors (over $\mathsf{U} \times \mathsf{U}$) through the object Eq of equivalences via $i = \mathsf{IdtoEq}$,

$$
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& & \mathsf{Eq} \arrow[d] \\
& & \mathsf{U} \times \mathsf{U} \arrow[d] \\
Z \arrow[rr] \arrow[urr, dotted, "k"] & & \mathsf{U}
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Z \arrow[r] \arrow[uur, dotted, "j"] \arrow[ur, dotted, "k"] & \mathrm{U}
\end{tikzcd}
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But the projection $\mathrm{Eq} \longrightarrow \mathrm{U}$ is a trivial fibration by EEP, so there is a diagonal filler j.

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\end{tikzcd}$$

But the projection $\mathrm{Eq} \longrightarrow \mathrm{U}$ is a trivial fibration by EEP, so there is a diagonal filler j. Composing gives the required k. $\square$

Done!

Done!

But is our QMS *right proper*?

Postscript: Frobenius

Definition (Frobenius)

The *Frobenius Property* says that trivial cofibrations pull back along fibrations,

$$\begin{tikzcd} A' \arrow[r] \arrow[d, rightarrowtail, "\sim"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, rightarrowtail, "\sim"] \\ A \arrow[r, twoheadrightarrow] & X . \end{tikzcd}$$

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It is equivalent to the condition that fibrations "push forward" along fibrations,

$$\begin{tikzcd} B \arrow[d, two heads] & \Pi_A B \arrow[d, two heads] \\ A \arrow[r, two heads] & X \end{tikzcd} .$$

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This is related to the existence of Π-types. It implies that our QMS is *right proper*.

Frobenius

Proposition

The Frobenius property holds for $(\mathsf{TCof}, \mathcal{F})$.

Proof.

$$
\begin{tikzcd}
B^{\mathbb{I}} \arrow[r, "\delta \Rightarrow B", "|" marking] \arrow[dr] & B^*_\epsilon \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B_\epsilon \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d] \\
 & A^{\mathbb{I}} \arrow[r, "\delta \Rightarrow A", "|" marking] \arrow[dr] & A_\epsilon \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d] \\
 & & X^{\mathbb{I}} \arrow[r, "\epsilon"'] & X
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& A^{\mathbb{I}} \arrow[r, "\delta \Rightarrow A", "|" marking] \arrow[dr] & A_\epsilon \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathbb{I}} \arrow[r, "\epsilon"'] & X \\
& (\Pi_A B)^{\mathbb{I}} \arrow[r, "\delta \Rightarrow \Pi_A B"'] \arrow[ur] & (\Pi_A B)_\epsilon \arrow[r] \arrow[u] & \Pi_A B \arrow[u]
\end{tikzcd}
$$

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The Frobenius property holds for $(\mathsf{TCof}, \mathcal{F})$.

Proof.

$$
\begin{tikzcd}
B^{\mathbb{I}} \arrow[r, "\delta\Rightarrow B", "\shortmid" marking] \arrow[dr] & B^*_\epsilon \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_\epsilon \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
 & A^{\mathbb{I}} \arrow[r, "\delta\Rightarrow A", "\shortmid" marking] \arrow[dr] & A_\epsilon \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
 & & X^{\mathbb{I}} \arrow[r, "\epsilon"'] & X \\
 & (\Pi_A B)^{\mathbb{I}} \arrow[ur] \arrow[r, "\delta\Rightarrow\Pi_A B"'] \arrow[d, dotted, bend right] & (\Pi_A B)_\epsilon \arrow[u] \arrow[r] \arrow[d, dotted, bend left] & \Pi_A B \arrow[u] \\
 & \Pi_{A^{\mathbb{I}}} B^{\mathbb{I}} \arrow[u] \arrow[r, "\shortmid" marking] & \Pi_{A^{\mathbb{I}}} B^*_\epsilon \arrow[u] &
\end{tikzcd}
$$

$\square$

That's all Folks!