# Informal cubical type theory

Bruno Bentzen

Carnegie Mellon University, USA
bbentzen@andrew.cmu.edu

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1. Informal type theory
   - Motivation
2. Cubical type theory
   - Higher cubes
   - The path type
   - Kan operations
3. Proofs
   - Groupoid operations
   - Weak connections
   - Groupoid laws
   - Path induction

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- the study of conventions for doing everyday mathematics in natural language assuming type theory as the underlying foundation.

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For homotopy type theory:

- the project was carried out in the HoTT book.

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For homotopy type theory:

- the project was carried out in the HoTT book.

Cubical type theory is more amenable to constructive interpretations, but it can be a challenge to understand for the uninitiated.

- the informal type theory project is a nice way to remedy the situation.

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"cubical type theory" means "cartesian cubical type theory" [ABC+17]

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Cubical type theory is based on the same basic homotopical perspective as homotopy type theory [AW09, Voe06] in which we regard

- a type $A$ as a space;
- a term $a : A$ as a point of the space $A$;
- a function $f : A \to B$ as a continuous map;
- a path $p : \mathrm{path}_A(a, b)$ as a path from point $a$ to $b$ in the space $A$;
- a type family $P : A \to \mathcal{U}$ as a fibration;
- $p : \mathrm{path}_{\mathcal{U}}(A, B)$ as a homotopy equivalence between spaces $A$ and $B$

By the homotopy hypothesis, homotopy types can be modelled by higher groupoids.

Higher groupoids can in turn be defined in terms of simplicial sets, or, as an alternative presentation, cubical sets.

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The main view that we take in this talk can be stated as:

*types in cubical type theory are cubical $\infty$-groupoids*

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We visualize a closed term $a : A$ as a point (0-cell).

$$\cdot\, a$$

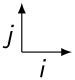
We think of an open term $p : A$ depending on $i : \mathbb{I}$ as a path (1-cell)

$$p[0/i] \xrightarrow{\quad p \quad} p[1/i] \qquad \xrightarrow[i]{}$$

with initial point $p[0/i]$ and terminal point $p[1/i]$.

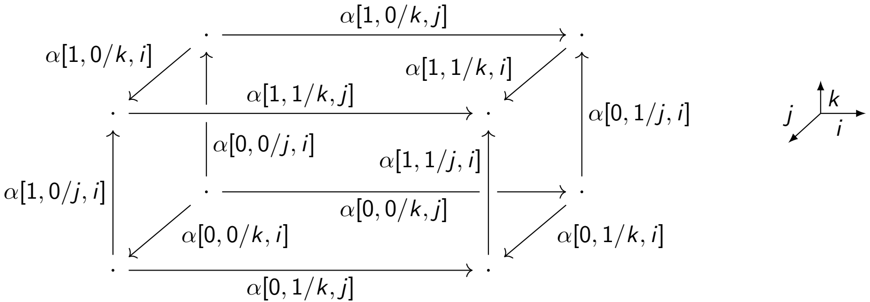
An open term $h : A$ depending on $i, j : \mathbb{I}$ is a "homotopy of paths" (2-cell)

$$\begin{array}{ccc}
h[1,0/j,i] & \xrightarrow{\quad h[1/j] \quad} & h[1,1/j,i] \\
{\scriptstyle h[0/i]}\uparrow & h & \uparrow{\scriptstyle h[1/i]} \\
h[0,0/j,i] & \xrightarrow[\quad h[0/j] \quad]{} & h[0,1/j,i]
\end{array}$$




where paths are allowed to have free (but path-connected) endpoints.

An open term $\alpha : A$ depending on $i, j, k : \mathbb{I}$ is a path over a path homotopy with free endpoints (3-cell)



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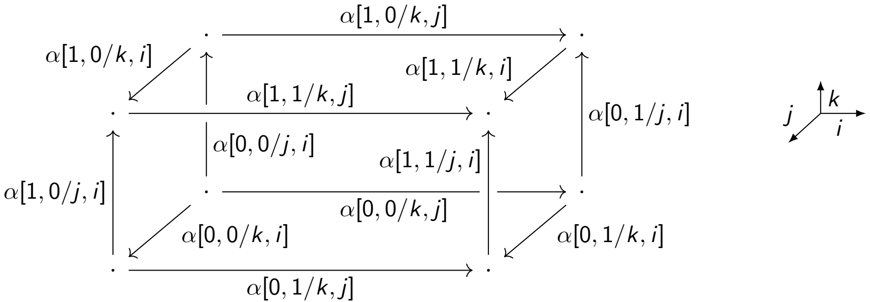


It is hard enough to visualize higher dimensions of space, but, most certainly, you can guess what comes next:

we think of higher-order open terms as higher-dimensional free path homotopies, and we picture them as hypercubes.

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Obvious choice: the type of functions from the interval, $\mathbb{I} \to A$ (line type).

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Obvious choice: the type of functions from the interval, $\mathbb{I} \to A$ (line type).

Given any type $A : \mathcal{U}$ and terms $a, b : A$ we can construct the type of paths from $a$ to $b$ in $A$, which we call their path type, denoted $\mathsf{path}_A(a, b)$. We explain the path type by prescribing:

- how to construct paths: abstraction ($\langle i \rangle p$)
- how can we use paths: application ($p@i$)
- what equalities they induce: $\alpha, \beta, \eta$ and, for $p : \mathsf{path}_A(a, b)$,

$$p@0 \equiv a : A \qquad\qquad p@1 \equiv b : A$$

(This is sometimes called the non-dependent path type)

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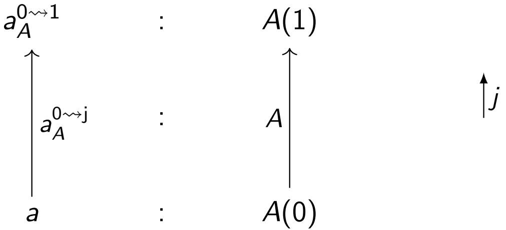
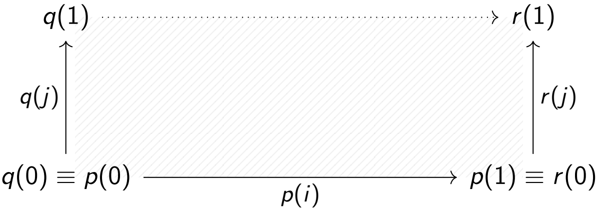
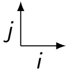
**Coercion**. This is a generalization of transport [The13, Lem 2.3.1]. Given $i, j : \mathbb{I}$, a path between types $A : \mathbb{I} \to \mathcal{U}$ and a term $a : A(i)$, there exists term of type $A(j)$, called the coercion of $a$ from $i$ to $j$ over $A$, and denoted by $a_A^{\mathrm{i} \rightsquigarrow \mathrm{j}} : A(j)$.




We also require that static coercions have no effect, i.e. $a_A^{\mathrm{i} \rightsquigarrow \mathrm{i}} \equiv a$.

**Composition**. This ensures that any open box can be filled.




This square is the filler of the composition and, for $i, j : \mathbb{I}$, it is denoted by

$$p(i)_A^{0 \rightsquigarrow j}[(i = 0) \mapsto j.q(j),\ (i = 1) \mapsto j.r(j)] : A$$

we insist that static compositions be ineffective, i.e.

$$p(i)_A^{k \rightsquigarrow k}[(i = 0) \mapsto j.q(j),\ (i = 1) \mapsto j.r(j)] \equiv p(i)$$

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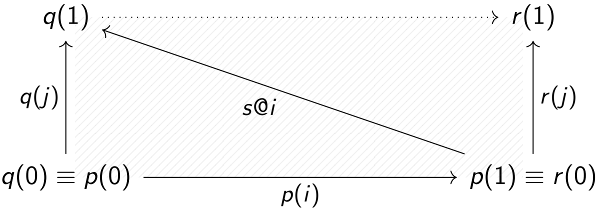


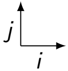


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What is a proof? Without being too philosophical about it:

> a proof is a sufficient argument for the truth of a proposition.

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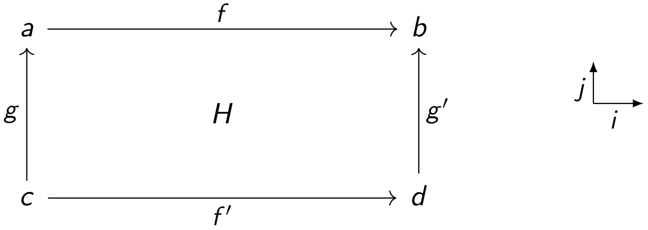
> a proof is a sufficient argument for the truth of a proposition.

Just as in category theory, we consider diagram chasing as a sufficient argument in (informal) cubical type theory.




Except that we understand commutative diagrams homotopically!

Let us start with identity, composition and inversion of morphisms:

**Lemma (Reflexivity)**

*For every type $A$ and every $a : A$, there exists a path*

$$\mathsf{path}_A(a, a)$$

*called the reflexivity path of $a$ and denoted* $\mathsf{refl}_a$.

Let us start with identity, composition and inversion of morphisms:

**Lemma (Reflexivity)**

*For every type $A$ and every $a : A$, there exists a path*

$$\mathsf{path}_A(a, a)$$

*called the reflexivity path of $a$ and denoted* $\mathsf{refl}_a$.

**Proof.**

Suppose that $i : \mathbb{I}$ is a fresh interval point. Since $a$ does not depend on $i$, meaning that $a[\epsilon/i] \equiv a$, for $\epsilon = 0, 1$, we have a degenerate line in the $i$ "direction" from $a$ to $a$ in $A$, and $\langle i \rangle a$ gives us the required path. □

## Lemma (Path inversion)

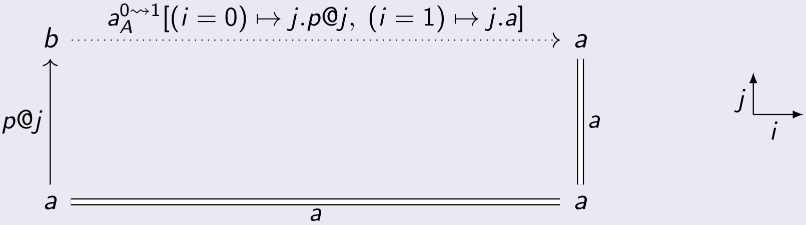
*For every type A and every* $a, b : A$*, there is a function*

$$\mathsf{path}_A(a, b) \to \mathsf{path}_A(b, a)$$

*called the inverse function and denoted* $p \mapsto p^{-1}$.

## Proof.

Note that $p : \text{path}_A(a, b)$ gives a "$j$-line" $p@j$ from $a$ to $b$ in $A$. We have an open box (where degeneracy is indicated using double bars):




By composition, it must have a lid, so, by path abstraction on the resulting (dotted) $i$-line, we have a path from $b$ to $a$ in $A$. □

## Lemma (Path concatenation)

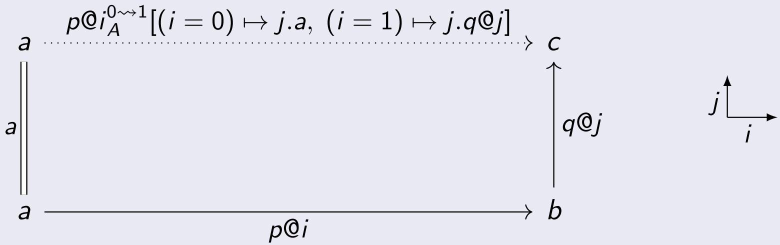
*For every type A and every* $a, b, c : A$*, there is a function*

$$\mathsf{path}_A(a, b) \to \mathsf{path}_A(b, c) \to \mathsf{path}_A(a, c)$$

*denoted* $p \mapsto q \mapsto p \cdot q$*. We call* $p \cdot q$ *the concatenation of* $p$ *and* $q$*.*

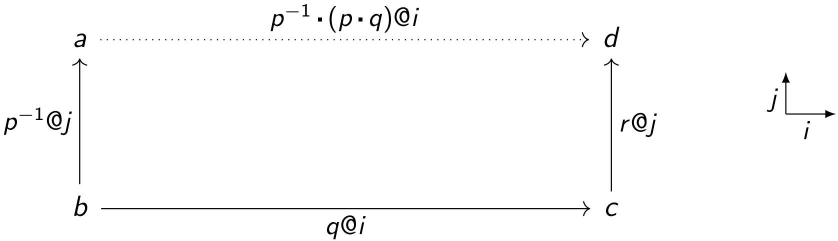
## Proof.

Given paths $p : \mathsf{path}_A(a, b)$ and $q : \mathsf{path}_A(b, c)$, we can construct an $i$-line $p@i$ from $a$ to $b$ and a $j$-line $q@j$ from $b$ to $c$. We have an open square:




We obtain the required path from $a$ to $c$ in $A$ by path abstraction on the line obtained by composition. □

The basic intuition:




*Composition generalizes path inversion and concatenation!*

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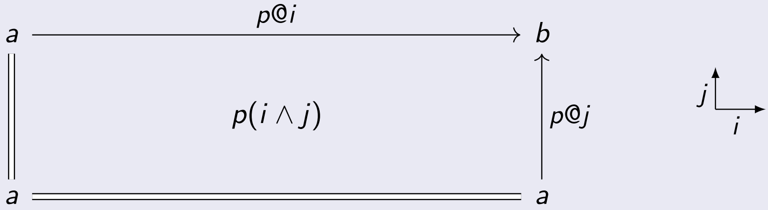
## Lemma (Meet)

*Suppose* $A : \mathcal{U}$, $a, b : A$ *and* $p : \mathrm{path}_A(a, b)$. *There is an operation*

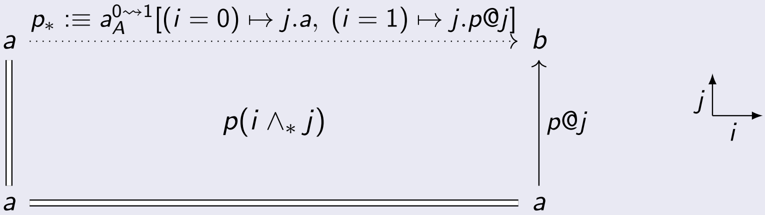
$$p(- \wedge -) : \mathbb{I} \to \mathbb{I} \to A$$

*such that, for any* $i, j : \mathbb{I}$, *the following holds:*



## Proof.

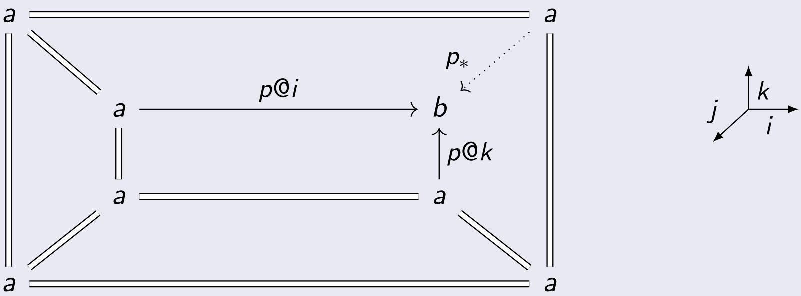
Given $p : \mathsf{path}_A(a, b)$, we are to find a $(i, j)$-square whose top face is $p@i$, right face is $p@j$, left and bottom faces are $a$. First, by composition, we obtain a "halfway" connection




and we use it to perform a two-extent composition

## Proof.

attaching it to the back and right faces of open cube




□

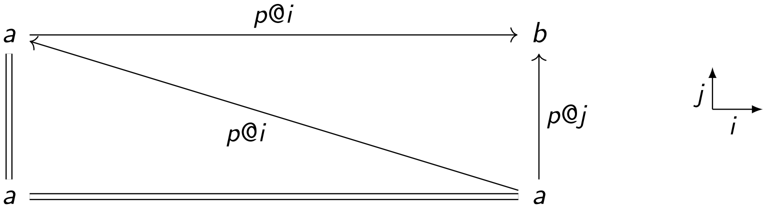
Moral of the story: two wrongs make a right!

Note that we could also have written out this proof in a full formal style:

$$\lambda A.\lambda a.\lambda b.\lambda p.\lambda i.\lambda k.$$

$$\begin{aligned}
& a_A^{0 \rightsquigarrow 1} \\
& \quad [(i = 0) \mapsto j.a, \\
& \quad\; (i = 1) \mapsto j.a_A^{0 \rightsquigarrow \mathrm{k}}[(j = 0) \mapsto k.a,\ (j = 1) \mapsto k.p@k], \\
& \quad\; (k = 0) \mapsto j.a_A^{0 \rightsquigarrow \mathrm{i}}[(j = 0) \mapsto i.a,\ (j = 1) \mapsto i.p@i], \\
& \quad\; (k = 1) \mapsto j.a] : \\
& \prod_{(A:\mathcal{U})} \prod_{(a,b:A)} \prod_{(p:\mathrm{path}_A(a,b))} \mathbb{I} \to \mathbb{I} \to A
\end{aligned}$$

Also, note that the proof could can be slightly improved



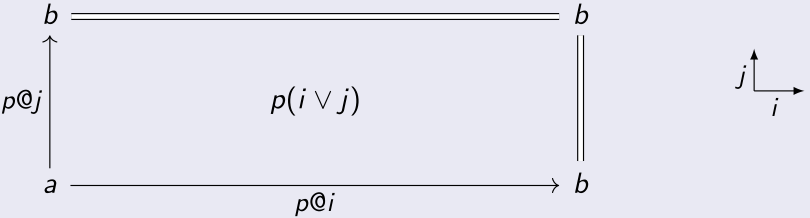

by making the diagonal definitionally equal to $p$!

## Lemma (Join)

*Given* $a, b : A$, *for any* $p : \mathrm{path}_A(a, b)$, *there is a function*

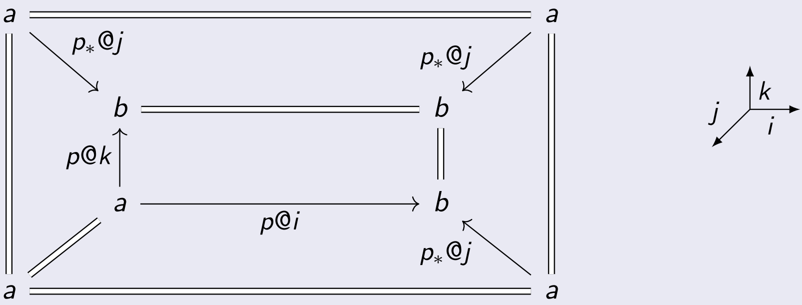
$$p(- \vee -) : \mathbb{I} \to \mathbb{I} \to A$$

*such that, for* $i, j : \mathbb{I}$, *we have:*



## Proof.

By composition using $p(- \wedge_* -)$:




□

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## Lemma (Right unit law)

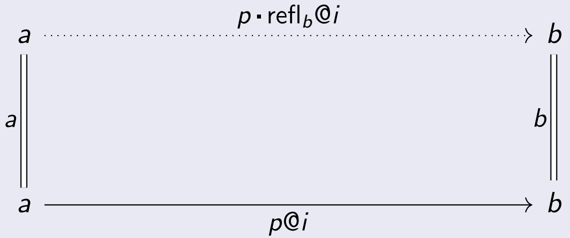
*For every* $A$ *and every* $a, b : A$ *we have a path*

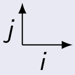
$$\mathsf{ru}_p : \mathsf{path}_{\mathsf{path}_A(a,b)}(p, p \cdot \mathsf{refl}_b)$$

*for any* $p : \mathsf{path}_A(a, b)$.

## Proof.

We need to construct a square:




but the filler of path concatenation already gives us this! □

## Lemma (Left unit law)

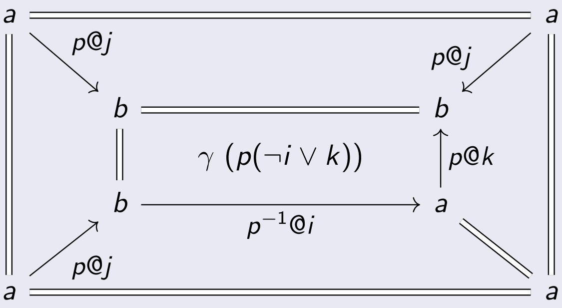
*For every A and every $a, b : A$ we have a path*

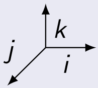
$$\mathsf{lu}_p : \mathsf{path}_{\mathsf{path}_A(a,b)}(p, \mathsf{refl}_a \cdot p)$$

*for any* $p : \mathsf{path}_A(a, b)$.

## Proof.

By composition, we define a helper $(i, j)$-square that goes from $p^{-1}@i$ to $b$ in the $i$-direction and from $b$ to $p@j$ in the $j$-direction.

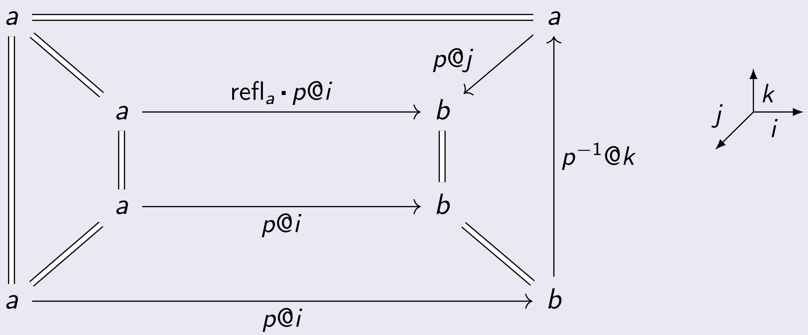



We use the filler of the path inversion of $p$ (bottom), meet (right), degenerate lines (back) and points (left and bottom).

## Proof.

Forming a new open cube, we set $\gamma$ at the right, the filler of $\mathrm{refl}_a \cdot p$ at the back, the filler of $p^{-1}$ at the bottom, and degenerate squares at the other faces.




□

### Lemma (Right cancellation)

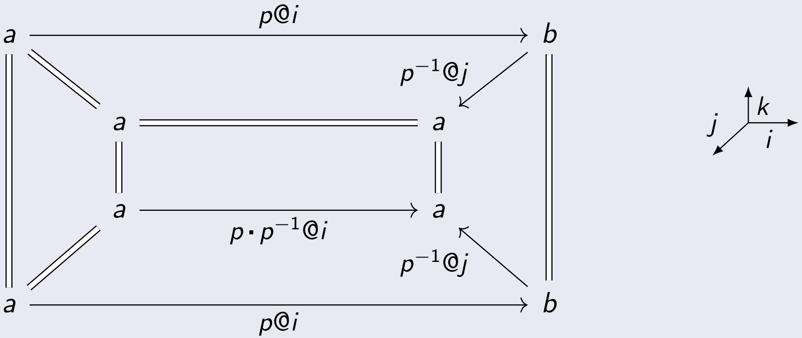
*For every A and every* $a, b : A$ *we have a path*

$$\mathsf{rc}_p : \mathsf{path}_{\mathsf{path}_A(a,b)}(\mathsf{refl}_a, p \cdot p^{-1})$$

*for any* $p : \mathsf{path}_A(a, b)$.

## Proof.

We fill the following open $(i, j, k)$-cube




whose back and front squares are respectively the fillers for path inversion and concatenation. □

## Lemma (Left cancellation)

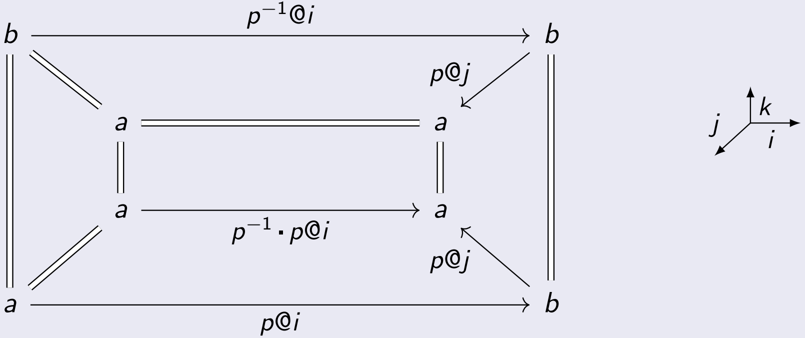
*For every A and every* $a, b : A$ *we have a path*

$$\mathsf{lc}_p : \mathsf{path}_{\mathsf{path}_A(b,b)}(\mathsf{refl}_b, p^{-1} \cdot p)$$

*for any* $p : \mathsf{path}_A(a, b)$.

## Proof.

By composition on the following open cube, whose back face is the $\gamma$ square from the proof of Left Unit lemma.




□

## Lemma (Inversion involution)

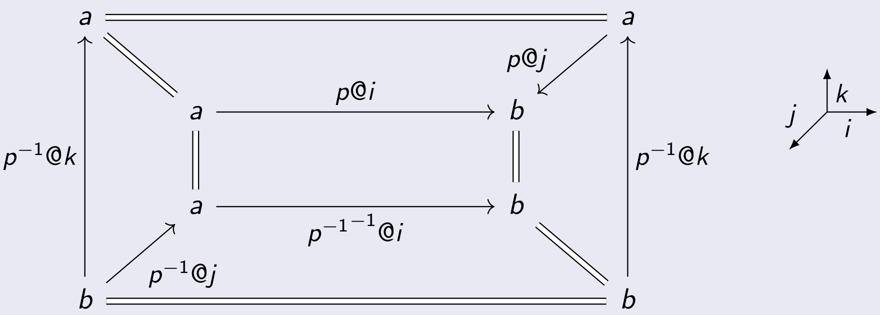
*For every A and every* $a, b : A$*, we have a path*

$$\mathsf{inv}_p : \mathsf{path}_{\mathsf{path}_A(a,b)}(p, {p^{-1}}^{-1})$$

*for any* $p : \mathsf{path}_A(a, b)$*.*

## Proof.

The proof follows by the use of meets, joins and $\gamma$ to form the composite:




□

## Lemma (Associativity)

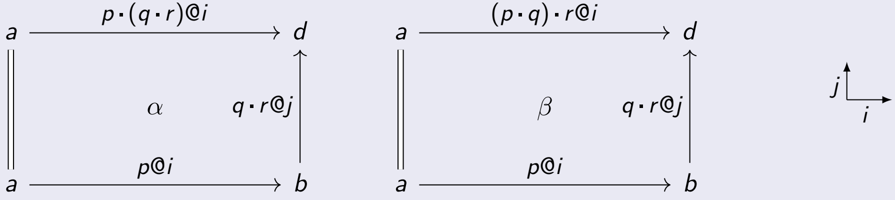
*For every A and every* $a, b, c, d : A$*, we have a path*

$$\mathsf{assoc}_{p,q,r} : \mathsf{path}_{\mathsf{path}_A(a,d)}((p \cdot q) \cdot r, p \cdot (q \cdot r))$$

*for any* $p : \mathsf{path}_A(a, b)$, $q : \mathsf{path}_A(a, b)$, $\mathsf{path}_A(c, d)$.

## Proof.

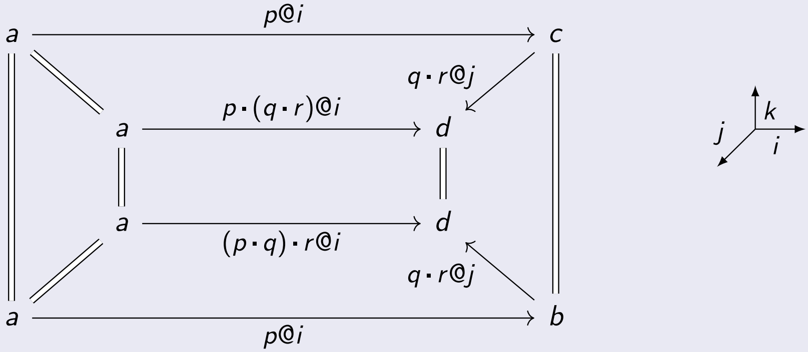
The proof idea is that any two squares with definitionally equal bottom, right and left faces must have the same top up to a path.




where leftmost square is just the filler of path concatenation

## Proof.

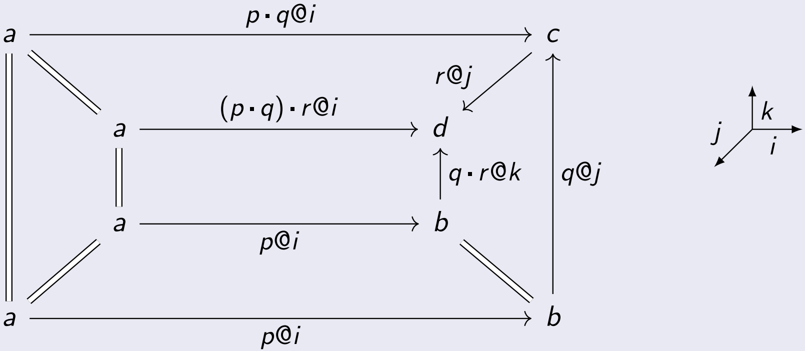
The identification can be derived by a simple composition:




Now we just have to construct the rightmost square.

## Proof.

It can be obtained by composition on the filler of path concatenation.

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### Theorem (Based path induction)

*Given $A : \mathcal{U}$, $a : A$, and a type family $C : \prod_{(x:A)} \mathsf{path}_A(a,x) \to \mathcal{U}$, we have a function*

$$\mathsf{pathrec} : \prod_{(x:A)} \prod_{(p:\mathsf{path}_A(a,x))} \prod_{(u:C(a,\mathsf{refl}_a))} C(x,p).$$

## Proof.

We want to construct, for every $x : A$, $p : \mathsf{path}_A(a, x)$, and $u : C(a, \mathsf{refl}_a)$, a term of $C(x, p)$. We shall use coercion on $u$ over a type line $C' : \mathbb{I} \to \mathcal{U}$ between $C'(0) :\equiv C(a, \mathsf{refl}_a)$ and $C'(1) :\equiv C(x, p)$.

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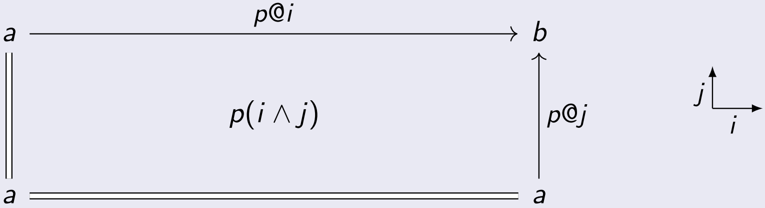
We use the meet square of $p$:




This square induces the desired type line

$$C' :\equiv \lambda i. C(p@i, \langle j \rangle p(i \wedge j)) : \mathbb{I} \to \mathcal{U}$$

## Proof.

Because $C'$ goes from

$$\begin{aligned} C'(0) &\equiv C(p@0, \langle j \rangle p(0 \wedge j)) \\ &\equiv C(a, \langle j \rangle a) \end{aligned}$$

to

$$\begin{aligned} C'(1) &\equiv C(p@1, \langle j \rangle p(1 \wedge j)) \\ &\equiv C(x, \langle j \rangle (p@j)) \\ &\equiv C(x, p) \quad \text{(via } \eta\text{-rule)} \end{aligned}$$

Now we complete the proof by coercing $u : C'(0)$ from 0 to 1. □

# Thank you!

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