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Identity types as equality predicates

Reconciling hyperdoctrines with MLTT

HoTT 2019 – Carnegie Mellon University

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1. Lawvere's hyperdoctrines

2. Reconcile hyperdoctrines with intensional equalities



1. Lawvere's hyperdoctrines



Lawvere's hyperdoctrines

An **hyperdoctrine** is a pseudofunctor $P : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$ such that:

- \mathcal{C} has finite products,
- each $P(f)$ has both a left adjoint \exists_f and a right adjoint \forall_f ,
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What does it have to do with logic?

Seely's semantics is an hyperdoctrine

$$\begin{array}{ccc} & & X \\ & & \downarrow p \\ A & \xrightarrow{f} & B \end{array}$$

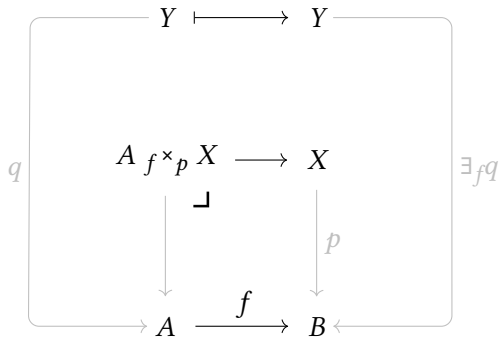
Seely's semantics is an hyperdoctrine

$$\begin{array}{ccc} A \times_p X & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow p \\ A & \xrightarrow{f} & B \end{array}$$

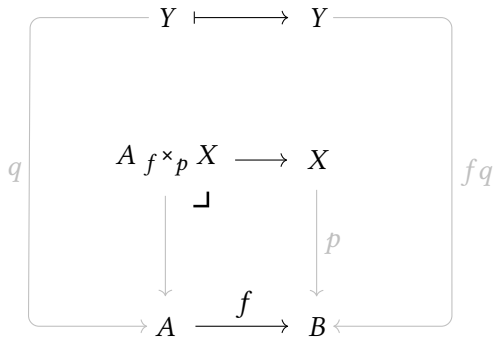
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$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ & & A \times_p X \longrightarrow X \\ & \lrcorner & \downarrow p \\ & & A \xrightarrow{f} B \\ & & \uparrow q \end{array}$$

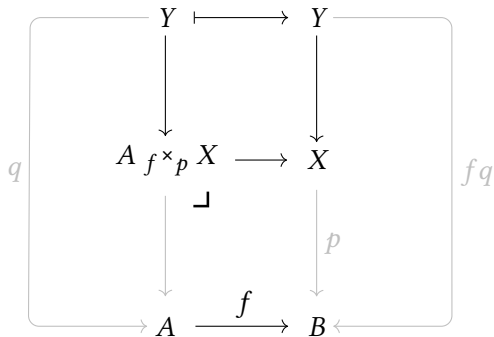
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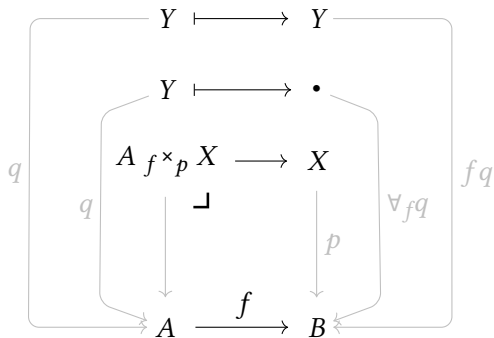
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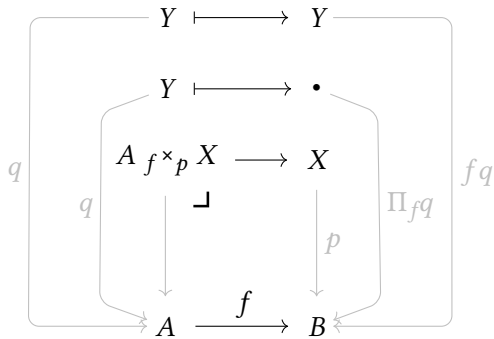
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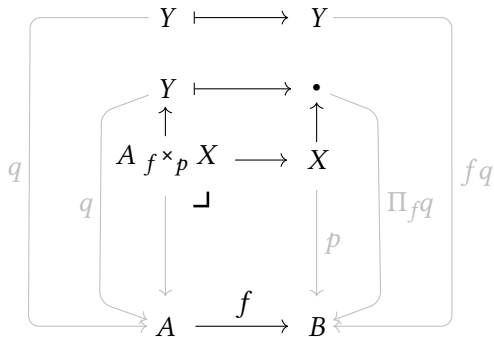
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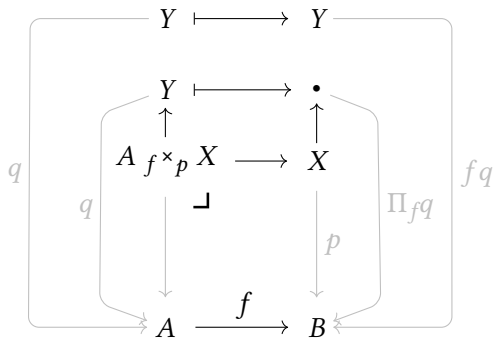
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Seely's semantics is an hyperdoctrine



In particular for $f = \delta_A : A \rightarrow A \times A$,

$$\exists_{\delta_A} : \text{id}_A \mapsto \delta_A$$

Subsets form an hyperdoctrine

$$\begin{array}{ccc} & & V \\ & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

Subsets form an hyperdoctrine

$$\begin{array}{ccc} f^{-1}(V) & \longleftarrow & V \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

Subsets form an hyperdoctrine

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Subsets form an hyperdoctrine

$$U \longmapsto \exists_f U$$

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Subsets form an hyperdoctrine

$$U \longmapsto f(U) = \{b \in B : \exists a \in A, f(a) = b \wedge a \in U\}$$

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$$U \longmapsto f(U)$$

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In particular for $f = \delta_A : A \rightarrow A \times A$,

$$\exists_{\delta_A} : A \mapsto \{(a, a') \in A \times A : a = a'\}$$

Predicates form an hyperdoctrine

$$\begin{array}{ccc} & & \psi(\vec{y}) \\ & & \vdots \\ \vec{x} & \xrightarrow{\vec{t}(\vec{x})} & \vec{y} \end{array}$$

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In particular for $\vec{t}(\vec{x}) = (\vec{x}, \vec{x}) : (x_1, \dots, x_n) \rightarrow (x_1, \dots, x_{2n})$,

$$\exists_{(\vec{x}, \vec{x})} : \top \mapsto \bigwedge_i x_i = x_{n+i}$$

Elementary existential doctrines

An **hyperdoctrine** is a pseudofunctor $P : \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$ such that:

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Define the **equality predicate** over $c \in \mathcal{C}$ as the direct image of $\mathbf{1}_c$ along the diagonal

$$\mathbf{1}_c \xrightarrow{\exists_{\Delta}} \exists_{\Delta}(\mathbf{1}_c)$$

$$c \xrightarrow{\Delta} c \times c$$

Grothendieck bifibrations

A Grothendieck *fibration* is a functor $p : \mathcal{E} \rightarrow \mathcal{B}$ such that

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$$\begin{array}{ccc} & & Y \\ & & \vdots \\ & & B \\ A & \xrightarrow{u} & B \end{array}$$

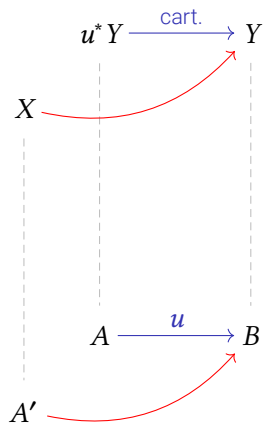
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A Grothendieck *fibration* is a functor $p : \mathcal{E} \rightarrow \mathcal{B}$ such that

$$\begin{array}{ccc} u^* Y & \xrightarrow{\text{cart.}} & Y \\ \vdots & & \vdots \\ A & \xrightarrow{u} & B \end{array}$$

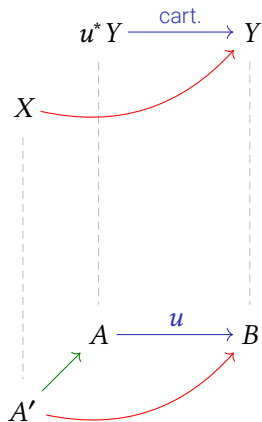
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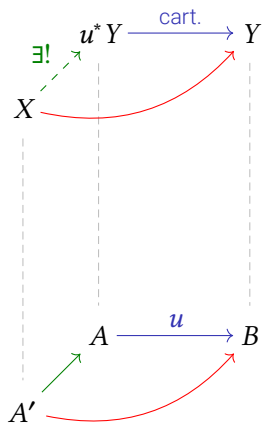
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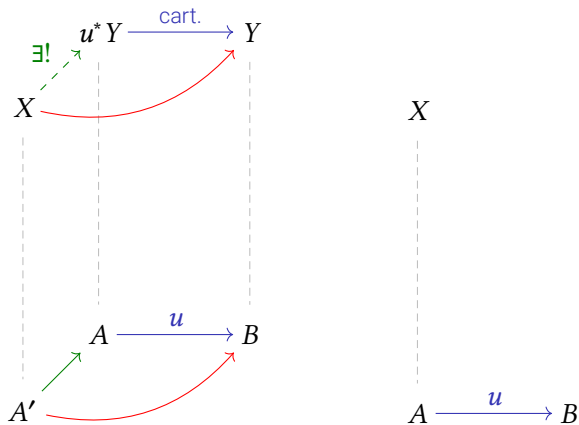
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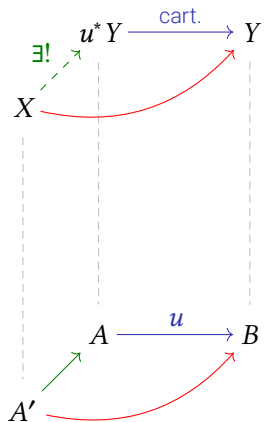
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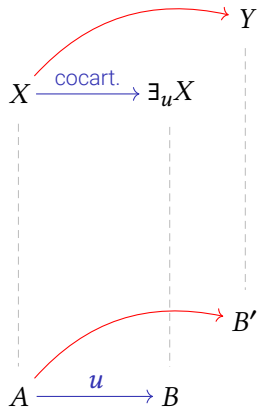
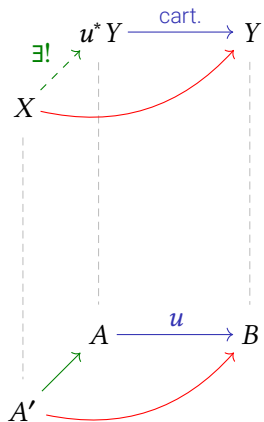
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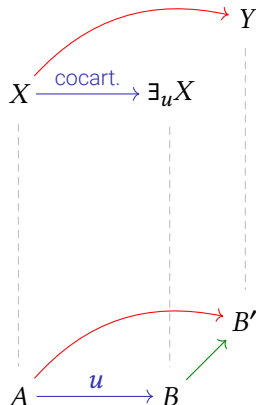
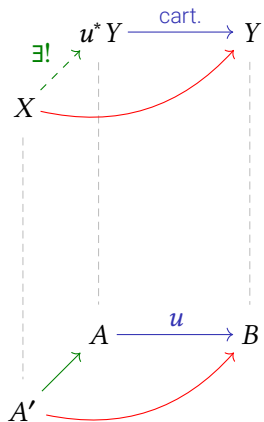
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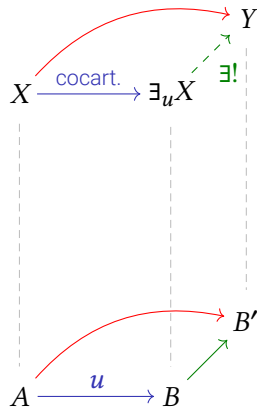
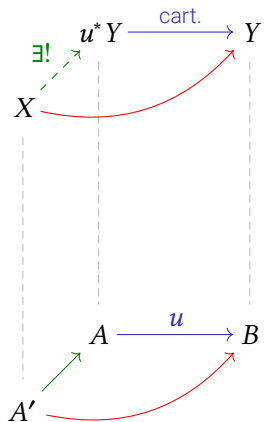
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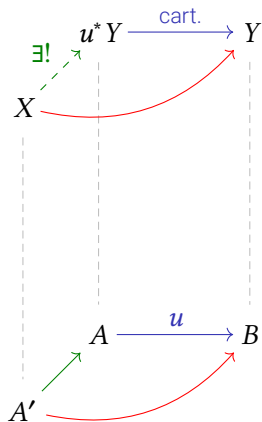
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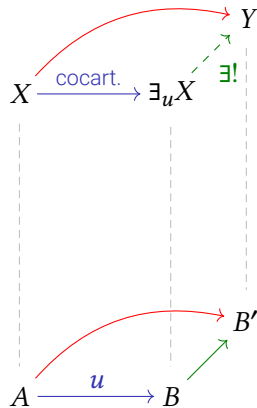


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Grothendieck construction

From an EED $P : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$, construct a Grothendieck bifibration:

$$\int_{\mathcal{C}} P$$
$$\downarrow$$
$$\mathcal{C}$$

- **objects:** pairs (X, A) for $X \in P(A)$
- **morphisms $(X, A) \rightarrow (Y, B)$:** pairs (u, f) where $u : A \rightarrow B$ and $f : X \rightarrow P(u)(Y)$.

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From a Grothendieck bifibration $p : \mathcal{E} \rightarrow \mathcal{B}$, construct:

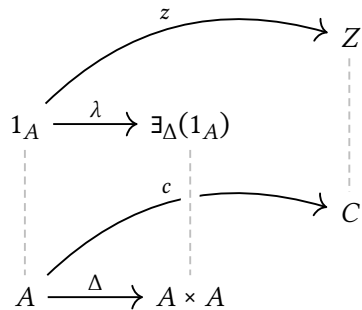
$$\mathcal{B}^{\text{op}} \xrightarrow{\tilde{p}} \mathbf{Cat}$$

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{E}_A \\ u \downarrow & & u^* \uparrow \\ B & \longrightarrow & \mathcal{E}_B \end{array} \left. \vphantom{\begin{array}{ccc} A & \longrightarrow & \mathcal{E}_A \\ u \downarrow & & u^* \uparrow \\ B & \longrightarrow & \mathcal{E}_B \end{array}} \right\} \exists_u$$

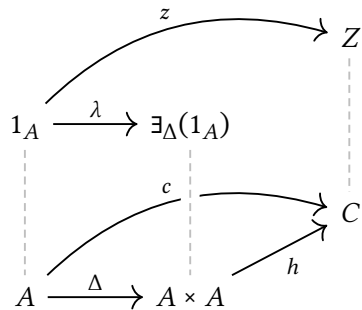
EEDs are extensional

$$\begin{array}{ccc} 1_A & \xrightarrow{\lambda} & \exists_{\Delta}(1_A) \\ \vdots & & \vdots \\ A & \xrightarrow{\Delta} & A \times A \end{array}$$

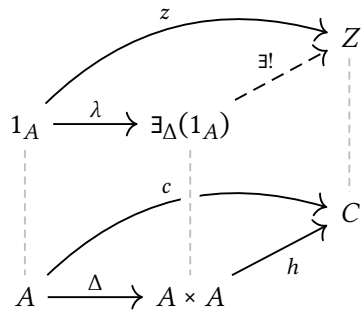
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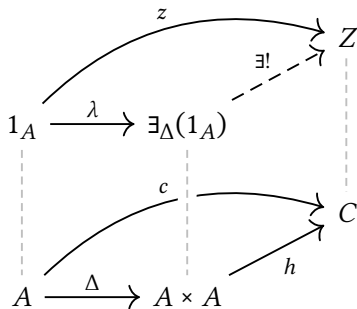


EEDs are extensional



EEDs are extensional

Equality in EEDs is intrinsically *extensional*.



2. Reconcile hyperdoctrines with intensional equalities



Primer on tribes

A **tribe** is a category \mathcal{C} with terminal object 1 and a class of maps \mathfrak{F} such that:

- $A \rightarrow 1$ is in \mathfrak{F} for every object A ,
- \mathfrak{F} contains every isomorphism,
- \mathfrak{F} is stable under change of base,
- \mathfrak{F} is stable under composition,
- $\mathfrak{F} \circ \text{LLP}(\mathfrak{F}) = \mathcal{C}$,
- $\text{LLP}(\mathfrak{F})$ is stable under change of base along elements of \mathfrak{F}

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Bare minimum to interpret a type theory with Σ , **Id**-types.

Goal

Provide a generalization of EEDs with

$$\text{cod} : \mathfrak{F} \rightarrow \mathcal{C}$$

as an instance.

Identity types in a tribe

Interpret \mathbf{Id}_A by factorizing:

$$\begin{array}{ccc} & & \mathbf{Id}_A \\ & \nearrow r_A & \downarrow p_A \\ A & \xrightarrow{\Delta} & A \times A \end{array}$$

$$\begin{array}{ccc} \bullet & & \bullet \\ \downarrow \epsilon_{\mathfrak{F}} & & \downarrow \text{LLP}(\mathfrak{F}) \\ \bullet & & \bullet \end{array}$$

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The *j-rule* is satisfied:

$$\begin{array}{ccc} A & \xrightarrow{c} & C \\ r_A \downarrow & & \downarrow \\ \text{Id}_A & \equiv & \text{Id}_A \end{array}$$

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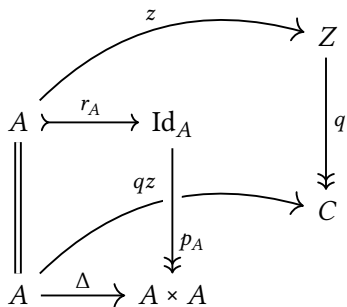
Identity types as an equality predicates?

The previous rules induces:

$$\begin{array}{ccc} A & \xrightarrow{r_A} & \text{Id}_A \\ \parallel & & \downarrow p_A \\ A & \xrightarrow{\Delta} & A \times A \end{array}$$

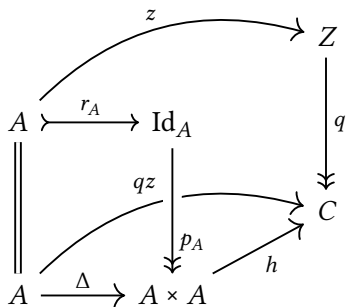
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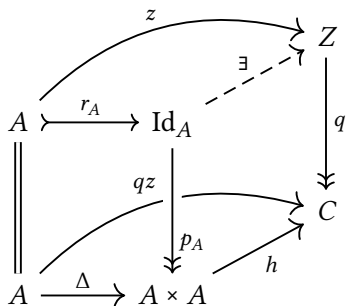
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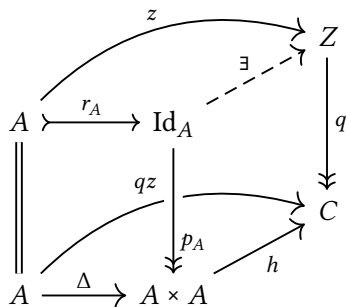
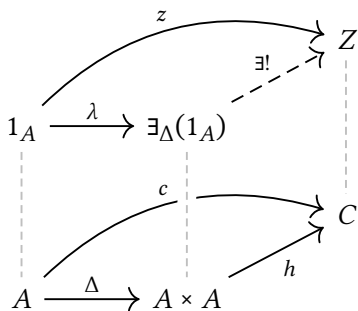
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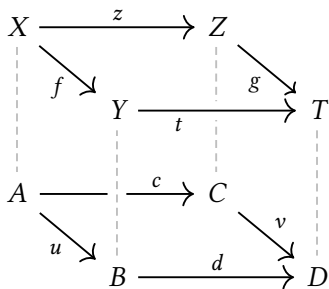
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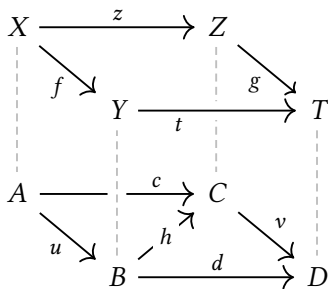
Relative lifting property

Given a functor $p : \mathcal{E} \rightarrow \mathcal{B}$, say that a map f in \mathcal{E} has the **weak left lifting property** relatively to p against g when:



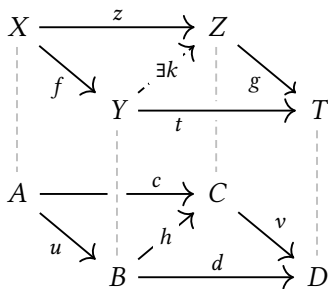
Relative lifting property

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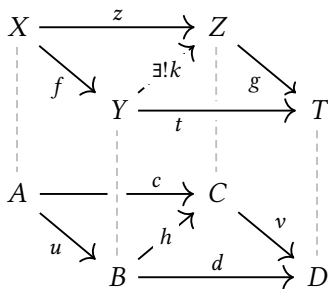
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Relative factorization systems

Define a **right weak factorization system** relative to $p : \mathcal{E} \rightarrow \mathcal{B}$ to consist of

- two classes $\mathcal{L}_{\mathcal{E}}, \mathcal{R}_{\mathcal{E}}$ of morphisms of \mathcal{E} ,
- and two classes $\mathcal{L}_{\mathcal{B}}, \mathcal{R}_{\mathcal{B}}$ of morphisms of \mathcal{B} ,

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- for every f in \mathcal{E} :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \vdots & & \vdots \\ A & \xrightarrow{u} & B \end{array}$$

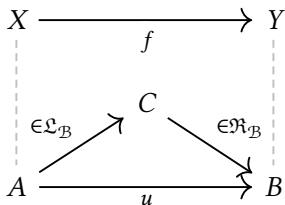
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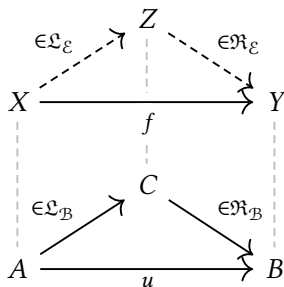
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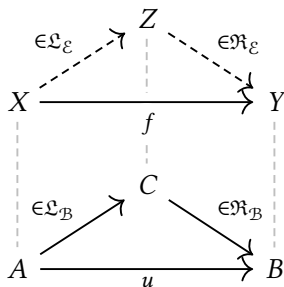
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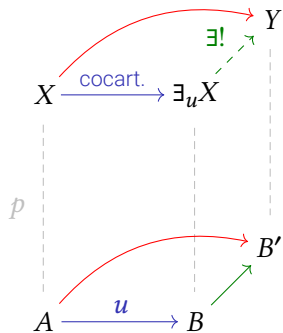
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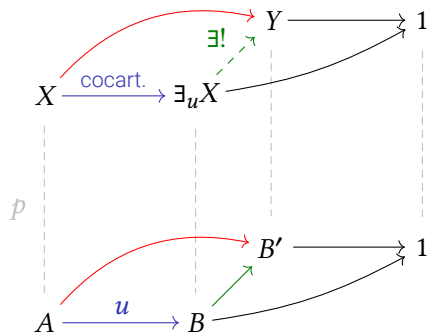
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Cocartesian morphism as lifting problems

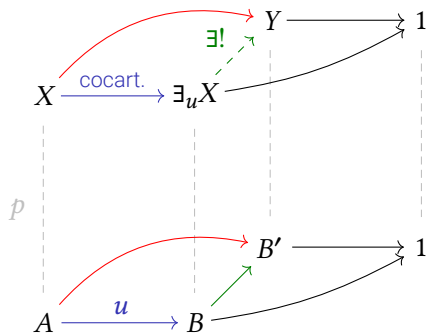


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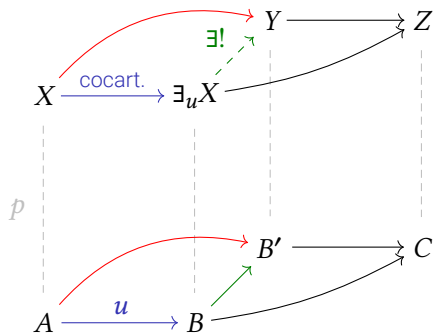
Cocartesian morphism as lifting problems

f is cocartesian if and only if $f \in \text{LLP}_p^\perp(\text{any} \rightarrow 1)$



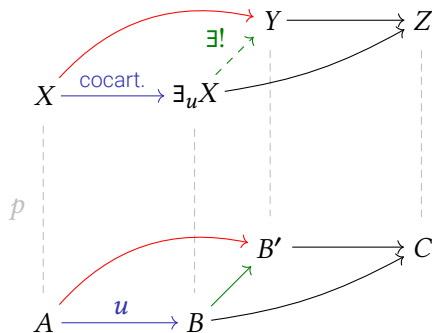
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f is cocartesian if and only if $f \in \text{LLP}_p^\perp(\text{Mor}(\mathcal{E}))$



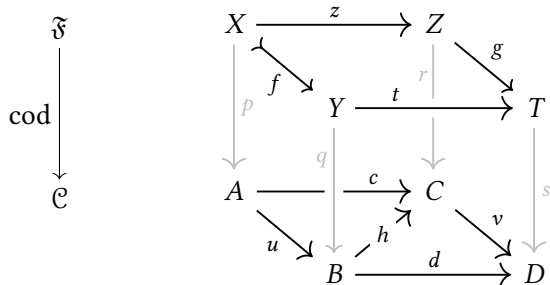
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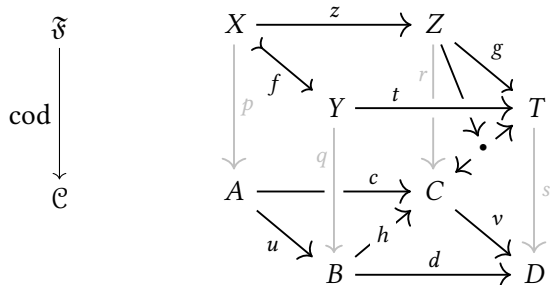


p is a Grothendieck opfibration if and only if there is a strong RFS relative to p with $\mathfrak{R}_{\mathcal{E}} = \text{Mor}(\mathcal{E})$ and $\mathfrak{L}_{\mathcal{B}} = \text{Mor}(\mathcal{B})$

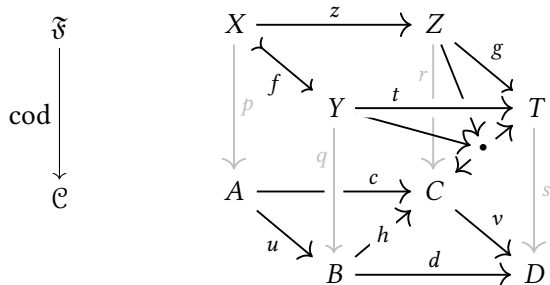
Anodyne maps as lifting problems



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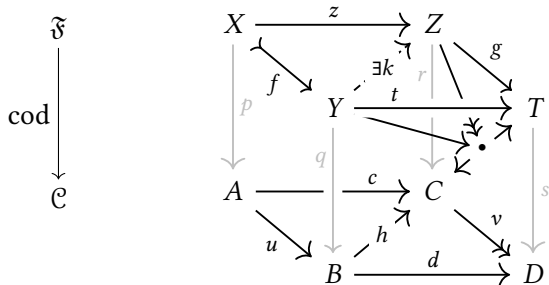


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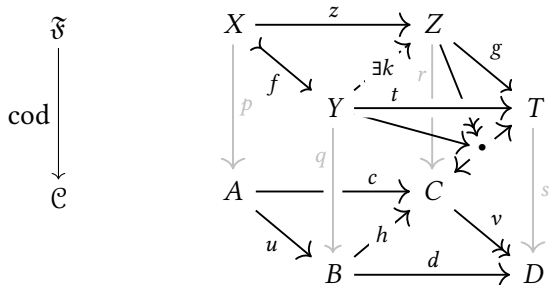
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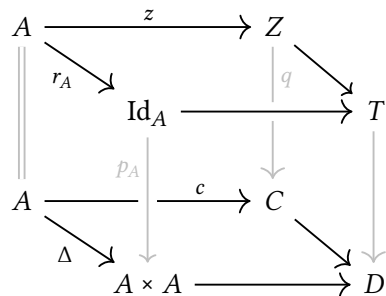
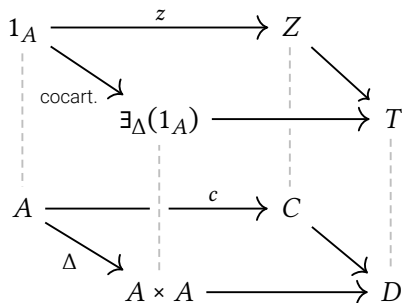
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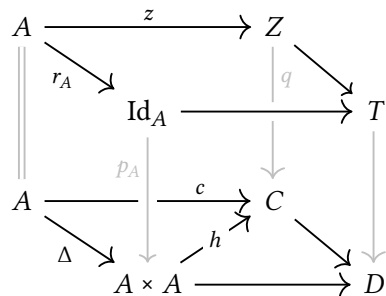
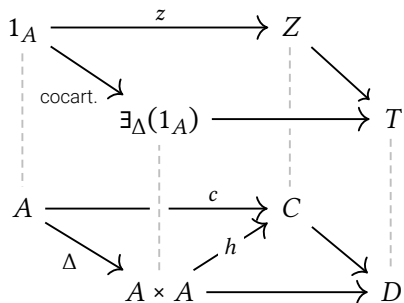


\mathcal{C} is a **tribe** if and only if there is a **weak RFS** relative to cod with $\mathfrak{R}_{\mathfrak{F}} = \{\text{Reedy fibrations}\}$ and $\mathfrak{L}_{\mathcal{C}} = \text{Mor}(\mathcal{C})$

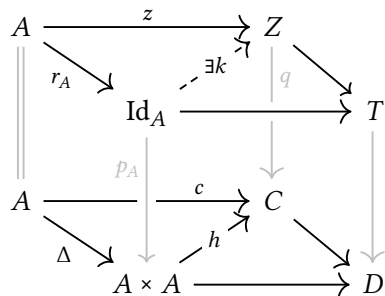
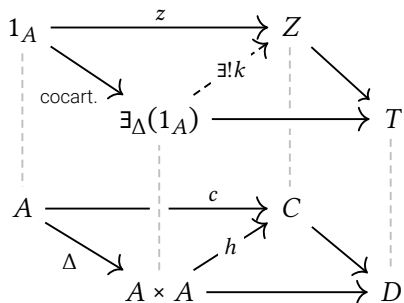
Reconcile Lawvere's equality and identity types



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Reconcile Lawvere's equality and identity types



Thank you.

<http://www.normalesup.org/~cagne/>

Consider the **groupoid hyperdoctrine**:

$$\mathbf{Grpd}^{\mathbf{Op}} \rightarrow \mathbf{CAT}$$

$$\mathcal{G} \mapsto \mathbf{Psh}(\mathcal{G})$$

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$$\begin{aligned}\mathbf{Grpd}^{\mathbf{Op}} &\rightarrow \mathbf{CAT} \\ \mathcal{G} &\mapsto \mathbf{Psh}(\mathcal{G})\end{aligned}$$

«

This should not to be taken as indicative of a lack of vitality of [the groupoid] hyperdoctrine, or even of a lack of a satisfactory theory of equality for it. Rather, it indicates that we have probably been too naive in defining equality in a manner too closely suggested by the classical conception.

»

— Lawvere

What is it good for?

A model M of a first-order theory \mathbb{T} can be interpreted as:

$$\begin{array}{ccc} \text{ctx}^{\text{op}} & \xrightarrow{\mathbb{T}} & \text{Cat} \\ \mathfrak{M} \downarrow & \mu \Downarrow & \uparrow \\ \text{Set}^{\text{op}} & \xrightarrow{\text{Sub}(-)} & \end{array}$$

where $\mathfrak{M} : \vec{x} \mapsto M^{|\vec{x}|}$, and $\mu_{\vec{x}} : \varphi(\vec{x}) \mapsto \{\vec{m} \mid M \models \varphi(\vec{m})\}$

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A P -model M of a first-order theory \mathbb{T} can be defined as:

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where \mathfrak{M} and μ have good properties.

Type-theoretic equality predicates

$$\frac{x: C \vdash Z(x) \text{ type} \quad x, y: A \vdash c(x, y): C \quad x: A \vdash z(x): Z(c(x, x))}{x, y: A, p: \text{Eq}_A(x, y) \vdash j_z(x, y, p): Z(c(x, y))}$$

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$$\frac{C \vdash Z \text{ type} \quad x, y: A \vdash c(x, y): C \quad x, y: A, p: \text{Eq}_A(x, y) \vdash k(x, y, p): Z(c(x, y))}{x, y: A, p: \text{Eq}_A(x, y) \vdash j_{k(x, x, \text{refl}_x)}(x, y, p) \equiv k(x, y, p)}$$