université PARIS DIDEROT PARIS 7

INSTITUT DE RECHERCHE EN INFORMATIQUE FONDAMENTALE

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Identity types as equality predicates

Reconciling hyperdoctrines with MLTT

HoTT 2019 – Carnegie Mellon University

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1. Lawvere's hyperdoctrines

2. Reconcile hyperdoctrines with intensional equalities

1. Lawvere’s hyperdoctrines

Lawvere's hyperdoctrines

An hyperdoctrine is a pseudofunctor $P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ such that:

- $\mathcal{C}$ has finite products,
- each $P(f)$ has both a left adjoint $\exists_f$ and a right adjoint $\forall_f$,
- each $P(c)$ is a cartesian closed category.

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What does it have to do with logic?

Seely's semantics is an hyperdoctrine

$$\begin{tikzcd} & X \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Seely's semantics is an hyperdoctrine

$$\begin{tikzcd}
A \,{}_f\!\times_p X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Seely's semantics is an hyperdoctrine

$$
\begin{tikzcd}
Y \arrow[dd, bend right, "q"'] & \\
A \,{}_f\!\times_p X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
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Seely's semantics is an hyperdoctrine

$$
\begin{tikzcd}
Y \arrow[r, mapsto] \arrow[dd, bend right, "q"'] & Y \arrow[dd, bend left, "\exists_f q"] \\
A \,{}_f\!\times_p X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
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\end{tikzcd}
$$

Seely's semantics is an hyperdoctrine

$$
\begin{tikzcd}
Y \arrow[r, mapsto] \arrow[dd, bend right=60, "q"'] & Y \arrow[dd, bend left=60, "fq"] \\
A \, {}_f\!\times_p X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
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Seely's semantics is an hyperdoctrine

$$
\begin{tikzcd}
Y \arrow[r, mapsto] \arrow[d] \arrow[dd, bend right, "q"'] & Y \arrow[d] \arrow[dd, bend left, "f\,q"] \\
A {}_{f}\!\times_{p} X \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "p"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
$$

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\begin{tikzcd}
Y \arrow[r, mapsto] \arrow[ddd, bend right=60, "q"'] & Y \arrow[ddd, bend left=60, "f\,q"] \\
Y \arrow[r, mapsto] \arrow[dd, bend right=40, "q"'] & \bullet \arrow[dd, bend left=40, "\forall_f q"] \\
A\ {}_f{\times}_p\ X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
A \arrow[r, "f"] & B
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Y \arrow[r, mapsto] \arrow[dd, bend right=40, "q"'] & \bullet \arrow[dd, bend left=40, "\Pi_f q"] \\
A \;{}_f\!\times_p X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
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A \,{}_f\!\times_p X \arrow[r] \arrow[u] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[u] \arrow[d, "p"] \\
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A \; {}_f\!\times_p \; X \arrow[u] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[u] \arrow[d, "p"] \\
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\end{tikzcd}$$

In particular for $f = \delta_A : A \longrightarrow A \times A$,

$$\exists_{\delta_A} : \mathrm{id}_A \longmapsto \delta_A$$

Subsets form an hyperdoctrine

$$\begin{tikzcd} & V \arrow[d, tail] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Subsets form an hyperdoctrine

$$\begin{tikzcd}
f^{-1}(V) \arrow[d, tail] & V \arrow[l, mapsto] \arrow[d, tail] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Subsets form an hyperdoctrine

U

$$\begin{tikzcd} f^{-1}(V) \arrow[d, tail] & V \arrow[l, maps to] \arrow[d, tail] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Subsets form an hyperdoctrine

$$U \longmapsto \exists_f U$$

$$\begin{tikzcd}
f^{-1}(V) \arrow[d, tail] & V \arrow[l, mapsto] \arrow[d, tail] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Subsets form an hyperdoctrine

$$U \longmapsto f(U) = \{b \in B : \exists a \in A, f(a) = b \wedge a \in U\}$$

$$\begin{tikzcd}
f^{-1}(V) \arrow[d, tail] & V \arrow[l, mapsto] \arrow[d, tail] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Subsets form an hyperdoctrine

$$
\begin{tikzcd}
U \arrow[r, mapsto] \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] & f(U) \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] = \{b \in B \,:\, \exists a \in A, f(a) = b \wedge a \in U\} \\
f^{-1}(V) \arrow[d, tail] & V \arrow[l, mapsto] \arrow[d, tail] \\
A \arrow[r, "f"] & B
\end{tikzcd}
$$

Subsets form an hyperdoctrine

$$U \longmapsto f(U)$$

$$U \longmapsto \forall_f U$$

$$\begin{tikzcd}
f^{-1}(V) \arrow[d, tail] & V \arrow[l, mapsto] \arrow[d, tail] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Subsets form an hyperdoctrine

$$U \longmapsto f(U)$$

$$U \longmapsto f_*(U) = \{b \in B : \forall a \in A, f(a) = b \Longrightarrow a \in U\}$$

$$\begin{tikzcd}
f^{-1}(V) \arrow[d, tail] & V \arrow[l, mapsto] \arrow[d, tail] \\
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Subsets form an hyperdoctrine

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U \arrow[r, mapsto] & f(U) & \\
U \arrow[r, mapsto] & f_*(U) & = \{b \in B \,:\, \forall a \in A, f(a) = b \Longrightarrow a \in U\} \\[-2em]
\rotatebox[origin=c]{90}{$\subseteq$} & \rotatebox[origin=c]{90}{$\subseteq$} & \\[-2em]
f^{-1}(V) \arrow[d, tail] & V \arrow[l, mapsto] \arrow[d, tail] & \\
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In particular for $f = \delta_A : A \longrightarrow A \times A$,

$$\exists_{\delta_A} : A \longmapsto \{(a, a') \in A \times A : a = a'\}$$

Predicates form an hyperdoctrine

$$\begin{tikzcd}
 & \psi(\vec{y}) \arrow[d, dash, dashed] \\
\vec{x} \arrow[r, "\vec{t}(\vec{x})"] & \vec{y}
\end{tikzcd}$$

Predicates form an hyperdoctrine

$$\begin{tikzcd}
\psi(\vec{t}(\vec{x})) \arrow[d, dash, dashed] & \psi(\vec{y}) \arrow[l, mapsto] \arrow[d, dash, dashed] \\
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\end{tikzcd}$$

Predicates form an hyperdoctrine

$$\varphi(\vec{x})$$

$$\begin{tikzcd}
\psi(\vec{t}(\vec{x})) \arrow[d, dashed, no head] & \psi(\vec{y}) \arrow[l, mapsto] \arrow[d, dashed, no head] \\
\vec{x} \arrow[r, "\vec{t}(\vec{x})"] & \vec{y}
\end{tikzcd}$$

Predicates form an hyperdoctrine

$$\varphi(\vec{x}) \longmapsto \exists \vec{x}, \left(\bigwedge_i t_i(\vec{x}) = y_i\right) \wedge \varphi(\vec{x})$$

$$\begin{tikzcd}
\psi(\vec{t}(\vec{x})) \arrow[d, dashed, no head] & \psi(\vec{y}) \arrow[l, mapsto] \arrow[d, dashed, no head] \\
\vec{x} \arrow[r, "\vec{t}(\vec{x})"] & \vec{y}
\end{tikzcd}$$

Predicates form an hyperdoctrine

$$
\begin{tikzcd}
\varphi(\vec{x}) \arrow[r, mapsto] \arrow[d, phantom, "\top"] & \exists \vec{x}, \left(\bigwedge_i t_i(\vec{x}) = y_i\right) \wedge \varphi(\vec{x}) \arrow[d, phantom, "\top"] \\
\psi(\vec{t}(\vec{x})) \arrow[d, dash, dashed] & \psi(\vec{y}) \arrow[l, mapsto] \arrow[d, dash, dashed] \\
\vec{x} \arrow[r, "\vec{t}(\vec{x})"] & \vec{y}
\end{tikzcd}
$$

Predicates form an hyperdoctrine

$$\varphi(\vec{x}) \longmapsto \exists \vec{x}, \left(\bigwedge_i t_i(\vec{x}) = y_i\right) \wedge \varphi(\vec{x})$$

$$\varphi(\vec{x}) \longmapsto \forall \vec{x}, \left(\bigwedge_i t_i(\vec{x}) = y_i\right) \Longrightarrow \varphi(\vec{x})$$

$$\begin{tikzcd}
\psi(\vec{t}(\vec{x})) \arrow[d, dashed, no head] & \psi(\vec{y}) \arrow[l, mapsto] \arrow[d, dashed, no head] \\
\vec{x} \arrow[r, "\vec{t}(\vec{x})"] & \vec{y}
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Predicates form an hyperdoctrine

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\begin{tikzcd}
\varphi(\vec{x}) \arrow[r, mapsto] & \exists \vec{x}, \left(\bigwedge_i t_i(\vec{x}) = y_i\right) \land \varphi(\vec{x}) \\
\varphi(\vec{x}) \arrow[r, mapsto] \arrow[d, phantom, "\perp"] & \forall \vec{x}, \left(\bigwedge_i t_i(\vec{x}) = y_i\right) \implies \varphi(\vec{x}) \arrow[d, phantom, "\perp"] \\
\psi(\vec{t}(\vec{x})) \arrow[d, dash, dashed] & \psi(\vec{y}) \arrow[l, mapsto] \arrow[d, dash, dashed] \\
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\vec{x} \arrow[r, "\vec{t}(\vec{x})"] & \vec{y}
\end{tikzcd}
$$

In particular for $\vec{t}(\vec{x}) = (\vec{x}, \vec{x}) : (x_1, \ldots, x_n) \longrightarrow (x_1, \ldots, x_{2n})$,

$$\exists_{(\vec{x},\vec{x})} : \top \mapsto \bigwedge_i x_i = x_{n+i}$$

Elementary existential doctrines

An hyperdoctrine is a pseudofunctor $P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ such that:

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- each $P(c)$ is a category with final object $\mathbf{1}_c$.

Define the equality predicate over $c \in \mathcal{C}$ as the direct image of $\mathbf{1}_c$ along the diagonal

$$\mathbf{1}_c \xmapsto{\exists_\Delta} \exists_\Delta(\mathbf{1}_c)$$

$$c \xrightarrow{\Delta} c \times c$$

Grothendieck bifibrations

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A Grothendieck fibration is a functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ such that

$$\begin{tikzcd}
u^* Y \arrow[r, "\text{cart.}"] \arrow[d, dash, dashed] & Y \arrow[d, dash, dashed] \\
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X \arrow[rrru, bend right] \arrow[ddd, dashed, no head] & & & \\
 & A \arrow[rr, "u"] & & B \\
A' \arrow[rrru, bend right] & & &
\end{tikzcd}$$

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$$
\begin{tikzcd}
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X \arrow[ur, dashed, "\exists!"] \arrow[urrr, bend right] \arrow[dd, dash, dashed] & & & \\
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\end{tikzcd}
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 & A \arrow[r, "u"] & B \\
A' \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "\text{cocart.}"] \arrow[d, dash, dashed] & \exists_u X \arrow[d, dash, dashed] \\
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& & B' \\
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 & & B' \\
A \arrow[r, "u"] \arrow[urr, bend left] & B \arrow[ur] &
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$$\begin{tikzcd}
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X \arrow[ur, dashed, "\exists!"] \arrow[urr, bend right] \arrow[dd, dash, dashed] & & \\
 & A \arrow[r, "u"] & B \\
A' \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & Y \arrow[dd, dash, dashed] \\
X \arrow[r, "\text{cocart.}"] \arrow[urr, bend left] \arrow[dd, dash, dashed] & \exists_u X \arrow[ur, dashed, "\exists!"'] \arrow[dd, dash, dashed] & \\
 & & B' \\
A \arrow[r, "u"] \arrow[urr, bend left] & B \arrow[ur] &
\end{tikzcd}$$

Grothendieck construction

From an EED $P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$, construct a Grothendieck bifibration:

$$\begin{array}{c} \int_{\mathcal{C}} P \\ \big\downarrow \\ \mathcal{C} \end{array}$$

- objects: pairs (X, A) for $X \in P(A)$
- morphisms $(X, A) \longrightarrow (Y, B)$: pairs (u, f) where $u : A \longrightarrow B$ and $f : X \longrightarrow P(u)(Y)$.

Grothendieck construction

From an EED $P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$, construct a Grothendieck bifibration:

$$\begin{tikzcd} \int_{\mathcal{C}} P \arrow[d] \\ \mathcal{C} \end{tikzcd}$$

- objects: pairs (X, A) for $X \in P(A)$
- morphisms $(X, A) \longrightarrow (Y, B)$: pairs (u, f) where $u : A \longrightarrow B$ and $f : X \longrightarrow P(u)(Y)$.

From a Grothendieck bifibration $p : \mathcal{E} \longrightarrow \mathcal{B}$, construct:

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{\tilde{p}} \mathbf{Cat}$$

$$\begin{tikzcd} A \arrow[r, mapsto] \arrow[d, "u"'] & \mathcal{E}_A \arrow[d, bend left, "\exists_u"] \\ B \arrow[r, mapsto] & \mathcal{E}_B \arrow[u, "u^*"] \arrow[u, phantom, shift left=3, "\dashv"] \end{tikzcd}$$

EEDs are extensional

$$\begin{tikzcd}
1_A \arrow[r, "\lambda"] \arrow[d, dash, dashed] & \exists_\Delta(1_A) \arrow[d, dash, dashed] \\
A \arrow[r, "\Delta"] & A \times A
\end{tikzcd}$$

EEDs are extensional

$$
\begin{tikzcd}
& & Z \arrow[dd, dash, dashed] \\
1_A \arrow[urr, bend left, "z"] \arrow[r, "\lambda"] \arrow[dd, dash, dashed] & \exists_\Delta(1_A) \arrow[dd, dash, dashed] & \\
& & C \\
A \arrow[urr, bend left, "c"] \arrow[r, "\Delta"] & A \times A &
\end{tikzcd}
$$

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 & & C \\
A \arrow[r, "\Delta"] \arrow[urr, bend left, "c"] & A \times A \arrow[ur, "h"'] &
\end{tikzcd}
$$

EEDs are extensional

$$
\begin{tikzcd}
 & & Z \arrow[dd, dash, dashed, gray] \\
1_A \arrow[r, "\lambda"] \arrow[urr, bend left, "z"] \arrow[dd, dash, dashed, gray] & \exists_\Delta(1_A) \arrow[ur, dashed, "\exists!"] \arrow[dd, dash, dashed, gray] & \\
 & & C \\
A \arrow[r, "\Delta"] \arrow[urr, bend left, "c"] & A \times A \arrow[ur, "h"'] &
\end{tikzcd}
$$

EEDs are extensional

Equality in EEDs is intrinsically extensional.

$$
\begin{tikzcd}
 & & Z \arrow[dd, dash, dashed, gray] \\
1_A \arrow[r, "\lambda"] \arrow[urr, bend left, "z"] \arrow[dd, dash, dashed, gray] & \exists_\Delta(1_A) \arrow[ur, dashed, "\exists!"] \arrow[dd, dash, dashed, gray] & \\
 & & C \\
A \arrow[r, "\Delta"] \arrow[urr, bend left, "c"] & A \times A \arrow[ur, "h"'] &
\end{tikzcd}
$$

2. Reconcile hyperdoctrines with intensional equalities

Primer on tribes

A tribe is a category $\mathfrak{C}$ with terminal object 1 and a class of maps $\mathfrak{F}$ such that:

- $A \longrightarrow 1$ is in $\mathfrak{F}$ for every object A,
- $\mathfrak{F}$ contains every isomorphism,
- $\mathfrak{F}$ is stable under change of base,
- $\mathfrak{F}$ is stable under composition,
- $\mathfrak{F} \circ \mathrm{LLP}(\mathfrak{F}) = \mathfrak{C}$,
- $\mathrm{LLP}(\mathfrak{F})$ is stable under change of base along elements of $\mathfrak{F}$

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- $\mathsf{LLP}(\mathfrak{F})$ is stable under change of base along elements of $\mathfrak{F}$

Bare minimum to interpret a type theory with Σ, Id-types.

Towards hypertribes

Goal

Provide a generalization of EEDs with

$$\mathrm{cod} : \mathfrak{F} \longrightarrow \mathfrak{C}$$

as an instance.

Identity types in a tribe

Interpret Id_A by factorizing:

$$\begin{tikzcd} & \mathrm{Id}_A \arrow[d, two heads, "p_A"] \\ A \arrow[ur, tail, "r_A"] \arrow[r, "\Delta"'] & A \times A \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[d, two heads, "\in\mathfrak{F}"] & \bullet \arrow[d, tail, "\mathrm{LLP}(\mathfrak{F})"] \\ \bullet & \bullet \end{tikzcd}$$

Identity types in a tribe

Interpret Id_A by factorizing:

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\end{tikzcd}
\qquad
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\bullet \arrow[d, two heads, "\in \mathfrak{F}"] \\
\bullet
\end{tikzcd}
\quad
\begin{tikzcd}
\bullet \arrow[d, tail, "\mathrm{LLP}(\mathfrak{F})"] \\
\bullet
\end{tikzcd}
$$

The j-rule is satisfied:

$$
\begin{tikzcd}
A \arrow[r, "c"] \arrow[d, tail, "r_A"'] & C \arrow[d, two heads] \\
\mathrm{Id}_A \arrow[r, equal] & \mathrm{Id}_A
\end{tikzcd}
$$

Identity types in a tribe

Interpret Id_A by factorizing:

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\end{tikzcd}
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\bullet
\end{tikzcd}
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\bullet
\end{tikzcd}
$$

The j-rule is satisfied:

$$
\begin{tikzcd}
A \arrow[r, "c"] \arrow[d, tail, "r_A"'] & C \arrow[d, two heads] \\
\mathrm{Id}_A \arrow[ur, "j"] \arrow[r, equal] & \mathrm{Id}_A
\end{tikzcd}
$$

Identity types as an equality predicates?

The previous rules induces:

$$\begin{tikzcd} A \arrow[r, tail, "r_A"] \arrow[d, equal] & \mathrm{Id}_A \arrow[d, two heads, "p_A"] \\ A \arrow[r, "\Delta"] & A \times A \end{tikzcd}$$

Identity types as an equality predicates?

The previous rules induces:

$$
\begin{tikzcd}
 & & Z \arrow[dd, two heads, "q"] \\
A \arrow[r, tail, "r_A"] \arrow[urr, bend left, "z"] \arrow[dd, equal] & \mathrm{Id}_A \arrow[dd, two heads, "p_A"] & \\
 & & C \\
A \arrow[r, "\Delta"] \arrow[urr, bend left, "qz"] & A \times A &
\end{tikzcd}
$$

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 & & C \\
A \arrow[r, "\Delta"] \arrow[urr, bend left, "qz"] & A \times A \arrow[ur, "h"'] &
\end{tikzcd}
$$

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 & & Z \arrow[dd, two heads, "q"] \\
A \arrow[urr, bend left, "z"] \arrow[r, tail, "r_A"] \arrow[dd, equal] & \mathrm{Id}_A \arrow[ur, dashed, "\exists"] \arrow[dd, two heads, "p_A"] & \\
 & & C \\
A \arrow[urr, bend left, "qz"] \arrow[r, "\Delta"] & A \times A \arrow[ur, "h"'] &
\end{tikzcd}
$$

Identity types as an equality predicates?

The previous rules induces:

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\begin{tikzcd}
& & Z \arrow[dd, dash, dashed, gray] \\
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& & C \\
A \arrow[r, "\Delta"] \arrow[urr, bend left, "c"] & A \times A \arrow[ur, "h"'] &
\end{tikzcd}
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& & C \\
A \arrow[r, "\Delta"] \arrow[urr, bend left, "qz"] & A \times A \arrow[ur, "h"'] &
\end{tikzcd}
$$

Relative lifting property

Given a functor $p : \mathcal{E} \longrightarrow \mathcal{B}$, say that a map f in $\mathcal{E}$ has the weak left lifting property relatively to p against g when:

$$
\begin{tikzcd}
X \arrow[rr, "z"] \arrow[dr, "f"'] \arrow[dd, dashed, no head] & & Z \arrow[dr, "g"] \arrow[dd, dashed, no head] & \\
& Y \arrow[rr, "t"'] \arrow[dd, dashed, no head] & & T \arrow[dd, dashed, no head] \\
A \arrow[rr, "c"] \arrow[dr, "u"'] & & C \arrow[dr, "v"] & \\
& B \arrow[rr, "d"] & & D
\end{tikzcd}
$$

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& Y \arrow[rr, "t"'] \arrow[dd, dash, dashed] & & T \arrow[dd, dash, dashed] \\
A \arrow[rr, "c"] \arrow[dr, "u"'] & & C \arrow[dr, "v"] & \\
& B \arrow[rr, "d"] \arrow[ur, dashed, "h"] & & D
\end{tikzcd}
$$

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Given a functor $p : \mathcal{E} \longrightarrow \mathcal{B}$, say that a map f in $\mathcal{E}$ has the weak left lifting property relatively to p against g when:

$$
\begin{tikzcd}
X \arrow[rr, "z"] \arrow[dr, "f"'] \arrow[dd, dash, dashed] & & Z \arrow[dr, "g"] \arrow[dd, dash, dashed] & \\
& Y \arrow[rr, "t"'] \arrow[ur, dashed, "\exists k"] \arrow[dd, dash, dashed] & & T \arrow[dd, dash, dashed] \\
A \arrow[rr, "c"] \arrow[dr, "u"'] & & C \arrow[dr, "v"] & \\
& B \arrow[rr, "d"] \arrow[ur, "h"] & & D
\end{tikzcd}
$$

Relative lifting property

Given a functor $p : \mathcal{E} \longrightarrow \mathcal{B}$, say that a map f in $\mathcal{E}$ has the strong left lifting property relatively to p against g when:

$$
\begin{tikzcd}
X \arrow[rr, "z"] \arrow[dr, "f"'] \arrow[dd, dash, dashed, gray] & & Z \arrow[dr, "g"] \arrow[dd, dash, dashed, gray] & \\
& Y \arrow[rr, "t"'] \arrow[ur, dashed, "\exists! k"] \arrow[dd, dash, dashed, gray] & & T \arrow[dd, dash, dashed, gray] \\
A \arrow[rr, "c"] \arrow[dr, "u"'] & & C \arrow[dr, "v"] & \\
& B \arrow[rr, "d"] \arrow[ur, "h"] & & D
\end{tikzcd}
$$

Relative factorization systems

Define a right weak factorization system relative to $p : \mathcal{E} \longrightarrow \mathcal{B}$ to consist of

- two classes $\mathfrak{L}_{\mathcal{E}}, \mathfrak{R}_{\mathcal{E}}$ of morphisms of $\mathcal{E}$,
- and two classes $\mathfrak{L}_{\mathcal{B}}, \mathfrak{R}_{\mathcal{B}}$ of morphisms of $\mathcal{B}$,

such that

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- and two classes $\mathfrak{L}_{\mathcal{B}}, \mathfrak{R}_{\mathcal{B}}$ of morphisms of $\mathcal{B}$,

such that

- $p(\mathfrak{L}_{\mathcal{E}}) \subseteq \mathfrak{L}_{\mathcal{B}}$ and $p(\mathfrak{R}_{\mathcal{E}}) \subseteq \mathfrak{R}_{\mathcal{B}}$,
- $\mathfrak{L}_{\mathcal{E}} = \mathsf{LLP}_p(\mathfrak{R}_{\mathcal{E}})$

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- and two classes $\mathfrak{L}_{\mathcal{B}}, \mathfrak{R}_{\mathcal{B}}$ of morphisms of $\mathcal{B}$,

such that

- $p(\mathfrak{L}_{\mathcal{E}}) \subseteq \mathfrak{L}_{\mathcal{B}}$ and $p(\mathfrak{R}_{\mathcal{E}}) \subseteq \mathfrak{R}_{\mathcal{B}}$,
- $\mathfrak{L}_{\mathcal{E}} = \mathsf{LLP}_p(\mathfrak{R}_{\mathcal{E}})$
- for every f in $\mathcal{E}$:

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d, dashed, no head] & Y \arrow[d, dashed, no head] \\ A \arrow[r, "u"'] & B \end{tikzcd}$$

Relative factorization systems

Define a right weak factorization system relative to $p : \mathcal{E} \longrightarrow \mathcal{B}$ to consist of

- two classes $\mathfrak{L}_{\mathcal{E}}, \mathfrak{R}_{\mathcal{E}}$ of morphisms of $\mathcal{E}$,
- and two classes $\mathfrak{L}_{\mathcal{B}}, \mathfrak{R}_{\mathcal{B}}$ of morphisms of $\mathcal{B}$,

such that

- $p(\mathfrak{L}_{\mathcal{E}}) \subseteq \mathfrak{L}_{\mathcal{B}}$ and $p(\mathfrak{R}_{\mathcal{E}}) \subseteq \mathfrak{R}_{\mathcal{B}}$,
- $\mathfrak{L}_{\mathcal{E}} = \mathsf{LLP}_p(\mathfrak{R}_{\mathcal{E}})$
- for every f in $\mathcal{E}$:

$$\begin{tikzcd}
X \arrow[rr, "f"'] \arrow[dd, dash, dashed] & & Y \arrow[dd, dash, dashed] \\
& C \arrow[dr, "\in\mathfrak{R}_{\mathcal{B}}"] & \\
A \arrow[ur, "\in\mathfrak{L}_{\mathcal{B}}"] \arrow[rr, "u"'] & & B
\end{tikzcd}$$

Relative factorization systems

Define a right weak factorization system relative to $p : \mathcal{E} \longrightarrow \mathcal{B}$ to consist of

- two classes $\mathfrak{L}_{\mathcal{E}}, \mathfrak{R}_{\mathcal{E}}$ of morphisms of $\mathcal{E}$,
- and two classes $\mathfrak{L}_{\mathcal{B}}, \mathfrak{R}_{\mathcal{B}}$ of morphisms of $\mathcal{B}$,

such that

- $p(\mathfrak{L}_{\mathcal{E}}) \subseteq \mathfrak{L}_{\mathcal{B}}$ and $p(\mathfrak{R}_{\mathcal{E}}) \subseteq \mathfrak{R}_{\mathcal{B}}$,
- $\mathfrak{L}_{\mathcal{E}} = \mathsf{LLP}_p(\mathfrak{R}_{\mathcal{E}})$
- for every f in $\mathcal{E}$:

$$\begin{tikzcd}
 & Z \arrow[dr, dashed, "\in\mathfrak{R}_{\mathcal{E}}"] \arrow[dd, dash, dashed] & \\
X \arrow[ur, dashed, "\in\mathfrak{L}_{\mathcal{E}}"] \arrow[rr, "f"'] \arrow[dd, dash, dashed] & & Y \arrow[dd, dash, dashed] \\
 & C \arrow[dr, "\in\mathfrak{R}_{\mathcal{B}}"] & \\
A \arrow[ur, "\in\mathfrak{L}_{\mathcal{B}}"] \arrow[rr, "u"'] & & B
\end{tikzcd}$$

Relative factorization systems

Define a right strong factorization system relative to $p : \mathcal{E} \longrightarrow \mathcal{B}$ to consist of

- two classes $\mathfrak{L}_\mathcal{E}, \mathfrak{R}_\mathcal{E}$ of morphisms of $\mathcal{E}$,
- and two classes $\mathfrak{L}_\mathcal{B}, \mathfrak{R}_\mathcal{B}$ of morphisms of $\mathcal{B}$,

such that

- $p(\mathfrak{L}_\mathcal{E}) \subseteq \mathfrak{L}_\mathcal{B}$ and $p(\mathfrak{R}_\mathcal{E}) \subseteq \mathfrak{R}_\mathcal{B}$,
- $\mathfrak{L}_\mathcal{E} = \mathsf{LLP}_p^{\perp}(\mathfrak{R}_\mathcal{E})$
- for every f in $\mathcal{E}$:

$$
\begin{tikzcd}
 & Z \arrow[dr, dashed, "\in\mathfrak{R}_\mathcal{E}"] \arrow[dd, dashed, no head, gray] & \\
X \arrow[ur, dashed, "\in\mathfrak{L}_\mathcal{E}"] \arrow[rr, "f"'] \arrow[dd, dashed, no head, gray] & & Y \arrow[dd, dashed, no head, gray] \\
 & C \arrow[dr, "\in\mathfrak{R}_\mathcal{B}"] & \\
A \arrow[ur, "\in\mathfrak{L}_\mathcal{B}"] \arrow[rr, "u"'] & & B
\end{tikzcd}
$$

Cocartesian morphism as lifting problems

$$
\begin{tikzcd}
 & & Y \arrow[dd, dash, dashed] \\
X \arrow[r, "\text{cocart.}"] \arrow[urr, bend left] \arrow[dd, dash, dashed, "p"'] & \exists_u X \arrow[ur, dashed, "\exists!"] \arrow[dd, dash, dashed] & \\
 & & B' \\
A \arrow[r, "u"] \arrow[urr, bend left] & B \arrow[ur] &
\end{tikzcd}
$$

Cocartesian morphism as lifting problems

$$
\begin{tikzcd}
 & & Y \arrow[r] \arrow[dd, dash, dashed] & 1 \arrow[dd, dash, dashed] \\
X \arrow[r, "\text{cocart.}"] \arrow[urr, bend left] \arrow[dd, dash, dashed, "p"'] & \exists_u X \arrow[ur, dashed, "\exists!"] \arrow[urr] \arrow[dd, dash, dashed] & & \\
 & & B' \arrow[r] & 1 \\
A \arrow[r, "u"] \arrow[urr, bend left] & B \arrow[ur] \arrow[urr] & &
\end{tikzcd}
$$

Cocartesian morphism as lifting problems

f is cocartesian if and only if $f \in \mathsf{LLP}_p^{\perp}(\text{any} \longrightarrow 1)$

$$
\begin{tikzcd}
 & & Y \arrow[r] \arrow[dd, dash, dashed] & 1 \arrow[dd, dash, dashed] \\
X \arrow[urr, bend left] \arrow[r, "\text{cocart.}"] \arrow[dd, dash, dashed, "p"'] & \exists_u X \arrow[ur, dashed, "\exists!"] \arrow[urr, bend right] \arrow[dd, dash, dashed] & & \\
 & & B' \arrow[r] & 1 \\
A \arrow[urr, bend left] \arrow[r, "u"] & B \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}
$$

Cocartesian morphism as lifting problems

f is cocartesian if and only if $f \in \mathsf{LLP}_p^{\perp}(\mathrm{Mor}\,(\mathcal{E}))$

$$
\begin{tikzcd}
 & & Y \arrow[r] \arrow[dd, dash, dashed] & Z \arrow[dd, dash, dashed] \\
X \arrow[urr, bend left] \arrow[r, "\text{cocart.}"] \arrow[dd, dash, dashed, "p"'] & \exists_u X \arrow[ur, dashed, "\exists!"] \arrow[urr] \arrow[dd, dash, dashed] & & \\
 & & B' \arrow[r] & C \\
A \arrow[urr, bend left] \arrow[r, "u"] & B \arrow[ur] \arrow[urr] & &
\end{tikzcd}
$$

Cocartesian morphism as lifting problems

f is cocartesian if and only if $f \in \mathsf{LLP}_p^{\perp}(\mathrm{Mor}\,(\mathcal{E}))$

$$\begin{tikzcd}
 & & Y \arrow[r] & Z \\
X \arrow[r, "\text{cocart.}"] \arrow[urr, bend left] \arrow[dd, dashed, no head, "p"'] & \exists_u X \arrow[ur, dashed, "\exists!"] \arrow[urr] \arrow[dd, dashed, no head] & & \\
 & & B' \arrow[r] \arrow[from=uu, dashed, no head] & C \arrow[from=uu, dashed, no head] \\
A \arrow[r, "u"] \arrow[urr, bend left] & B \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

p is a Grothendieck opfibration if and only if there is a strong RFS relative to p with $\mathfrak{R}_{\mathcal{E}} = \mathrm{Mor}\,(\mathcal{E})$ and $\mathfrak{L}_{\mathcal{B}} = \mathrm{Mor}\,(\mathcal{B})$

Anodyne maps as lifting problems

$$
\begin{tikzcd}
\mathfrak{F} \arrow[d, "\mathrm{cod}"'] \\
\mathcal{C}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[rr, "z"] \arrow[dr, tail, "f"'] \arrow[dd, "p"'] & & Z \arrow[dr, "g"] \arrow[dd, "r"'] & \\
& Y \arrow[rr, "t"] \arrow[dd, "q"'] & & T \arrow[dd, "s"] \\
A \arrow[rr, "c"] \arrow[dr, "u"'] & & C \arrow[dr, "v"] & \\
& B \arrow[rr, "d"] \arrow[ur, dashed, "h"] & & D
\end{tikzcd}
$$

Anodyne maps as lifting problems

$$
\begin{tikzcd}
\mathfrak{F} \arrow[d, "\mathrm{cod}"'] \\
\mathfrak{C}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[rr, "z"] \arrow[dr, tail, "f"'] \arrow[ddd, "p"'] & & Z \arrow[dr, "g"] \arrow[ddd, "r"] \arrow[ddr] & \\
& Y \arrow[rr, "t"] \arrow[ddd, "q"'] & & T \arrow[ddd, "s"] \\
& & & \bullet \arrow[u] \arrow[dl] \\
A \arrow[rr, "c"] \arrow[dr, "u"'] & & C \arrow[dr, "v"] & \\
& B \arrow[rr, "d"] \arrow[ur, dashed, "h"] & & D
\end{tikzcd}
$$

Anodyne maps as lifting problems

$$
\begin{tikzcd}
\mathfrak{F} \arrow[d, "\mathrm{cod}"'] \\
\mathfrak{C}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[rr, "z"] \arrow[dr, tail, "f"'] \arrow[ddd, "p"] & & Z \arrow[dr, "g"] \arrow[ddd, "r"'] \arrow[ddr] & \\
& Y \arrow[rr, "t"] \arrow[ddd, "q"'] \arrow[drr] & & T \arrow[ddd, "s"] \\
& & & \bullet \arrow[u, dashed] \arrow[dl] \\
A \arrow[rr, "c"] \arrow[dr, "u"'] & & C \arrow[dr, "v"] & \\
& B \arrow[ur, dashed, "h"] \arrow[rr, "d"] & & D
\end{tikzcd}
$$

Anodyne maps as lifting problems

f is anodyne if and only if $(f, u) \in \mathsf{LLP}_{\mathrm{cod}}(\text{Reedy fibrations})$

$$
\begin{tikzcd}
\mathfrak{F} \arrow[d, "\mathrm{cod}"'] \\
\mathcal{C}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[rr, "z"] \arrow[dr, tail, "f"'] \arrow[ddd, gray, "p"'] & & Z \arrow[dr, "g"] \arrow[ddr] \arrow[ddd, gray, "r"] & \\
& Y \arrow[rr, "t"] \arrow[rrd] \arrow[ur, dashed, "\exists k"] \arrow[ddd, gray, "q"'] & & T \arrow[ddd, gray, "s"] \\
& & & \bullet \arrow[u] \arrow[dl] \\
A \arrow[rr, "c"] \arrow[dr, "u"'] & & C \arrow[dr, "v"] & \\
& B \arrow[ur, "h"] \arrow[rr, "d"] & & D
\end{tikzcd}
$$

Anodyne maps as lifting problems

f is anodyne if and only if $(f, u) \in \mathbf{LLP}_{\mathrm{cod}}(\text{Reedy fibrations})$

$$\begin{tikzcd}
\mathfrak{F} \arrow[d, "\mathrm{cod}"'] \\
\mathfrak{C}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[rr, "z"] \arrow[dr, "f"'] \arrow[ddd, "p"'] & & Z \arrow[dr, "g"] \arrow[ddd, "r"] \arrow[ddr] & \\
& Y \arrow[rr, "t"] \arrow[ur, dashed, "\exists k"] \arrow[drr] \arrow[ddd, "q"'] & & T \arrow[ddd, "s"] \\
& & & \bullet \arrow[u] \arrow[dl] \\
A \arrow[rr, "c"] \arrow[dr, "u"'] & & C \arrow[dr, "v"] & \\
& B \arrow[ur, "h"] \arrow[rr, "d"] & & D
\end{tikzcd}$$

$\mathfrak{C}$ is a tribe if and only if there is a weak RFS relative to $\mathbf{cod}$ with $\mathfrak{R}_{\mathfrak{F}} = \{\text{Reedy fibrations}\}$ and $\mathfrak{L}_{\mathfrak{C}} = \mathrm{Mor}(\mathfrak{C})$

Reconcile Lawvere’s equality and identity types

$$
\begin{tikzcd}
1_A \arrow[rr, "z"] \arrow[dr, "\text{cocart.}"'] \arrow[dd, dashed, no head] & & Z \arrow[dr] \arrow[dd, dashed, no head] & \\
& \exists_\Delta(1_A) \arrow[rr] \arrow[dd, dashed, no head] & & T \arrow[dd, dashed, no head] \\
A \arrow[rr, "c"] \arrow[dr, "\Delta"'] & & C \arrow[dr] & \\
& A \times A \arrow[rr] & & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "z"] \arrow[dr, "r_A"'] \arrow[dd, equal] & & Z \arrow[dr] \arrow[dd, "q"] & \\
& \mathrm{Id}_A \arrow[rr] \arrow[dd, "p_A"'] & & T \arrow[dd] \\
A \arrow[rr, "c"] \arrow[dr, "\Delta"'] & & C \arrow[dr] & \\
& A \times A \arrow[rr] & & D
\end{tikzcd}
$$

Reconcile Lawvere's equality and identity types

$$
\begin{tikzcd}
1_A \arrow[rr, "z"] \arrow[dr, "\text{cocart.}"'] \arrow[dd, dash, dashed] & & Z \arrow[dr] \arrow[dd, dash, dashed] & \\
& \exists_\Delta(1_A) \arrow[rr] \arrow[dd, dash, dashed] & & T \arrow[dd, dash, dashed] \\
A \arrow[rr, "c"] \arrow[dr, "\Delta"'] & & C \arrow[dr] & \\
& A \times A \arrow[rr] \arrow[ur, "h"] & & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "z"] \arrow[dr, "r_A"'] \arrow[dd, equal] & & Z \arrow[dr] \arrow[dd, "q"] & \\
& \mathrm{Id}_A \arrow[rr] \arrow[dd, "p_A"'] & & T \arrow[dd] \\
A \arrow[rr, "c"] \arrow[dr, "\Delta"'] & & C \arrow[dr] & \\
& A \times A \arrow[rr] \arrow[ur, "h"] & & D
\end{tikzcd}
$$

Reconcile Lawvere's equality and identity types

$$\begin{tikzcd}
1_A \arrow[rr, "z"] \arrow[dr, "\text{cocart.}"'] \arrow[dd, dashed, no head] & & Z \arrow[dr] \arrow[dd, dashed, no head] & \\
& \exists_\Delta(1_A) \arrow[ur, dashed, "\exists! k"] \arrow[rr] \arrow[dd, dashed, no head] & & T \arrow[dd, dashed, no head] \\
A \arrow[rr, "c"] \arrow[dr, "\Delta"'] & & C \arrow[dr] & \\
& A \times A \arrow[ur, "h"] \arrow[rr] & & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "z"] \arrow[dr, "r_A"'] \arrow[dd, equal] & & Z \arrow[dr] \arrow[dd, "q"] & \\
& \mathrm{Id}_A \arrow[ur, dashed, "\exists k"] \arrow[rr] \arrow[dd, "p_A"'] & & T \arrow[dd] \\
A \arrow[rr, "c"] \arrow[dr, "\Delta"'] & & C \arrow[dr] & \\
& A \times A \arrow[ur, "h"] \arrow[rr] & & D
\end{tikzcd}$$

Thank you.

http://www.normalesup.org/~cagne/

Lawvere's insight

Consider the groupoid hyperdoctrine:

$$\begin{aligned} \mathsf{Grpd}^{\mathrm{op}} &\longrightarrow \mathsf{CAT} \\ \mathcal{G} &\longmapsto \mathrm{Psh}(\mathcal{G}) \end{aligned}$$

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> « This should not to be taken as indicative of a lack of vitality of [the groupoid] hyperdoctrine, or even of a lack of a satisfactory theory of equality for it. Rather, it indicates that we have probably been too naive in defining equality in a manner too closely suggested by the classical conception. »
>
> — *Lawvere*

What is it good for?

A model M of a first-order theory $\mathbb{T}$ can be interpreted as:

$$\begin{tikzcd} \mathsf{ctx}^{\mathrm{op}} \arrow[r, "\mathbb{T}"] \arrow[d, "\mathfrak{M}"'] & \mathsf{Cat} \arrow[dl, phantom, "\mu\Downarrow"] \\ \mathsf{Set}^{\mathrm{op}} \arrow[ur, bend right, "\mathsf{Sub}(-)"'] & \end{tikzcd}$$

where $\mathfrak{M} : \vec{x} \mapsto M^{|\vec{x}|}$, and $\mu_{\vec{x}} : \varphi(\vec{x}) \mapsto \{\vec{m} \mid M \vDash \varphi(\vec{m})\}$

What is it good for?

A P-model M of a first-order theory $\mathbb{T}$ can be defined as:

$$\begin{tikzcd} \mathsf{ctx}^{\mathrm{op}} \arrow[r, "\mathbb{T}"] \arrow[d, "\mathfrak{M}"'] \arrow[dr, phantom, "\mu\Downarrow"] & \mathsf{Cat} \\ \mathfrak{C}^{\mathrm{op}} \arrow[ur, bend right, "P"'] & \end{tikzcd}$$

where $\mathfrak{M}$ and μ have good properties.

Type-theoretic equality predicates

$$\frac{x : C \vdash Z(x)\ \text{type} \quad x, y : A \vdash c(x,y) : C \quad x : A \vdash z(x) : Z(c(x,x))}{x, y : A, p : \mathrm{Eq}_A(x,y) \vdash j_z(x,y,p) : Z(c(x,y))}$$

$$\frac{x : C \vdash Z(x)\ \text{type} \quad x, y : A \vdash c(x,y) : C \quad x : A \vdash z(x) : Z(c(x,x))}{x : A \vdash j_z(x,x,\mathsf{refl}_x) \equiv z(x)}$$

$$\frac{C \vdash Z\ \text{type} \quad x, y : A \vdash c(x,y) : C \quad x, y : A, p : \mathrm{Eq}_A(x,y) \vdash k(x,y,p) : Z(c(x,y)}{x, y : A, p : \mathrm{Eq}_A(x,y) \vdash j_{k(x,x,\mathsf{refl}_x)}(x,y,p) \equiv k(x,y,p)}$$