Localization in HoTT

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Outline:

- Motivation for localization
- Main results about p-localization
- Proofs and background results

Motivation for localization

Localization of spaces was developed by Adams, Bousfield, Dror, Mimura, Nishida, Quillen, Sullivan, Toda, etc., starting in the 1970s.

It is now a fundamental and pervasive tool in algebraic topology.

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There are many important theorems whose statement does not involve localization but which can be proved using localization. E.g.

Theorem (Serre). If Y is a simply connected, finite CW complex then either:

- Y is contractible, or
- $\pi_i Y$ is non-zero for infinitely many i.

Motivation for localization II

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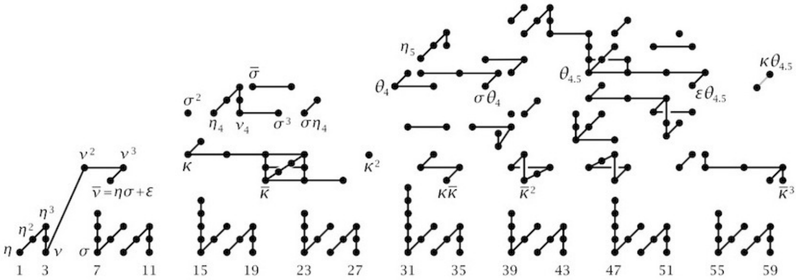
$p = 2:$



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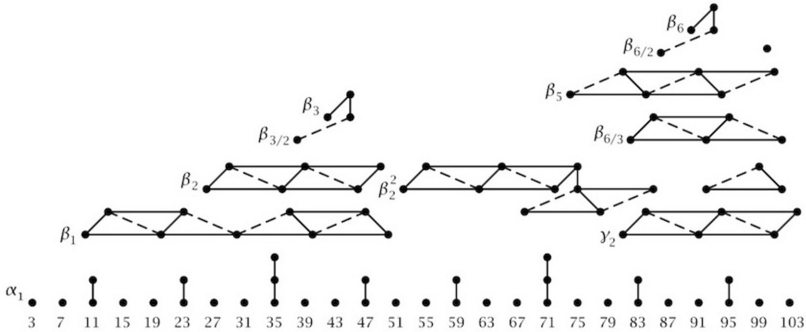

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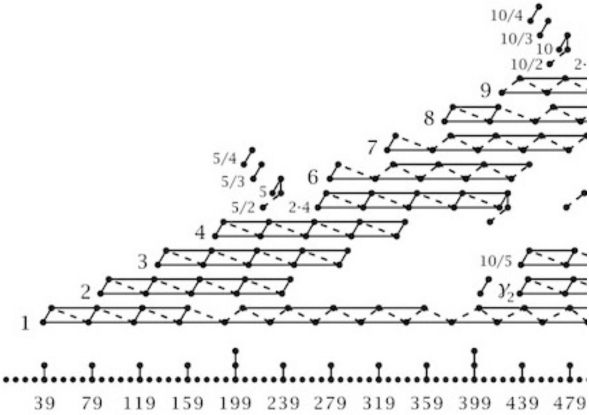
$p = 5:$



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To study such phenonmena, it's useful to replace the sphere with a "p-localized" version which only contains the p-primary part of the homotopy groups.

Many papers in algebraic topology start with the phrase "In this paper, we are working localized at a prime p" and then implicitly invoke localization technology throughout.

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Many computational techniques, such as the Adams spectral sequence, also work one prime at a time.

Motivation for localization III

A special case of localization is rationalization, which has the effect of tensoring all homotopy groups with $\mathbb{Q}$.

It turns out that the homotopy theory of rational spaces can be described completely algebraically (Quillen, Sullivan).
The algebraic description is very practical for computations.

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Using rationalization, one can prove:

Theorem (Serre). The groups $\pi_i(S^n)$ are all finite, except $\pi_n(S^n) \cong \mathbb{Z}$ and $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus$ finite.

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The work I'll describe brings localization into type theory, which is a necessary first step towards the results mentioned above.

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Fix a prime $p : \mathbb{N}$.

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Def. A type X is p-local if for every prime $q \neq p$ and every $x_0 : X$, the map

$$q : \Omega(X, x_0) \longrightarrow \Omega(X, x_0) \quad \text{sending} \quad \ell \longmapsto \ell^q$$

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Def. A p-localization of X is a p-local type $X_{(p)}$ and a map $\eta : X \to X_{(p)}$ such that for every p-local type Z, every map $X \to Z$ factors uniquely through $X \to X_{(p)}$.

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Theorem (Rijke, Shulman, Spitters). Every type X has a p-localization, unique up to equivalence, and functorial.

Main results

Theorem (CORS). For X simply connected, the natural map $\pi_n(X, x_0) \to \pi_n(X_{(p)}, \eta(x_0))$ is p-localization of abelian groups for every $n : \mathbb{N}$ and every $x_0 : X$.

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Theorem (Scoccola). Let R and S be denumerable sets of primes such that $R \cup S =$ all primes. Then, for X simply connected,

$$\begin{array}{ccc} X & \longrightarrow & X_{(R)} \\ \downarrow & & \downarrow \\ X_{(S)} & \longrightarrow & X_{(R\cap S)} \end{array}$$

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Scoccola has also developed the theory of nilpotent types, which can have non-trivial fundamental group, and has generalized the above results to such types. (For the second theorem, he needs to assume that X is truncated in this case.)

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In particular, if X is simply connected, then so is $X_{(p)}$.
The case $n = 1$ follows.

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Case $n > 1$: Consider the fiber sequence

$$K(\pi_{n+1}(X), n+1) \longrightarrow \left\|X\right\|_{n+1} \longrightarrow \left\|X\right\|_{n}.$$

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Applying p-localization gives

$$K(\pi_{n+1}(X), n+1)_{(p)} \longrightarrow \left\|X_{(p)}\right\|_{n+1} \longrightarrow \left\|X_{(p)}\right\|_{n},$$

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We'll show that this is again a fibre sequence and that the fibre is $K(\pi_{n+1}(X)_{(p)}, n+1)$, which will complete the proof.

p-Separated types

p-localization is not lex, i.e., it does not preserve all fibre sequences.
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Def. A type X is p-separated if for every $x, y : X$, the type $x = y$ is p-local.

Theorem (RSS). Every type X has a universal map $\eta' : X \to X'_{(p)}$ to a p-separated type.

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Theorem (RSS). Every type X has a universal map $\eta' : X \to X'_{(p)}$ to a p-separated type.

We prove:

Theorem (CORS). Any fibre sequence fits into a diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, "p\text{-equiv}"'] & E \arrow[r] \arrow[d, "\eta'"] & X \arrow[d, "\eta'"] \\ F' \arrow[r] & E'_{(p)} \arrow[r] & X'_{(p)}, \end{tikzcd}$$

where F' is the fibre of the bottom row and is therefore p-separated.

Proof, continued

Then we use:

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$$K(\pi_{n+1}(X), n+1) \longrightarrow \|X\|_{n+1} \longrightarrow \|X\|_n$$

are all simply connected, we get a fibre sequence

$$F' \longrightarrow \|X_{(p)}\|_{n+1} \longrightarrow \|X_{(p)}\|_n,$$

and a p-equivalence $K(\pi_{n+1}(X), n+1) \to F'$.

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(More generally, p-localization preserves fibre sequences of simply connected types. So, for X pointed and simply connected, $\Omega(X_{(p)}) \simeq (\Omega X)_{(p)}$.)

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(More generally, p-localization preserves fibre sequences of simply connected types. So, for X pointed and simply connected, $\Omega(X_{(p)}) \simeq (\Omega X)_{(p)}$.)

It remains to understand the p-localization of an Eilenberg-Mac Lane space.

Localizations of Eilenberg-Mac Lane spaces

Prop (CORS). For X pointed and simply connected, the natural map

$$\Omega X \longrightarrow \mathrm{colim}(\Omega X \xrightarrow{k_1} \Omega X \xrightarrow{k_2} \cdots)$$

is the p-localization of ΩX, where k_i is the product of the first i primes, excluding p.

Proof. It's not too hard to see that the map is a p-equivalence.

To see that it is p-local uses the compactness of S^1, which uses the work of van Doorn, Rijke and Sojakova on the identity types of sequential colimits. □

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Cor (CORS). For G abelian and $n \geq 1$, the p-localization of $K(G, n)$ is $K(G_{(p)}, n)$, where $G_{(p)}$ is the p-localization of G as an abelian group.

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Cor (CORS). For G abelian and $n \geq 1$, the p-localization of $K(G, n)$ is $K(G_{(p)}, n)$, where $G_{(p)}$ is the p-localization of G as an abelian group.

It follows that $\pi_n(X) \to \pi_n(X_{(p)})$ is p-localization.

Localization and truncation commute

We used:

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Lemma 1. The n-truncation of a p-local type is p-local.

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Lemma 2. The p-localization of a simply connected n-truncated type is n-truncated.

Indeed, the natural maps

$$X \to X_{(p)} \to \left\|X_{(p)}\right\|_n$$

and

$$X \to \left\|X\right\|_n \to (\left\|X\right\|_n)_{(p)}$$

are both universal maps to types that are both n-truncated and p-local.

Proof of Lemma 1

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Proof. This follows from the commutative diagram

$$\begin{tikzcd}
\Omega\|X\|_n \arrow[d, "q"'] & \|\Omega X\|_{n-1} \arrow[l, "\sim"'] \arrow[d, "\|q\|_{n-1}", "\sim"'] \\
\Omega\|X\|_n & \|\Omega X\|_{n-1}. \arrow[l, "\sim"']
\end{tikzcd}$$

where q is a prime different from p. □

Proof of Lemma 2

Lemma 2. The p-localization of a simply connected n-truncated type X is n-truncated.

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Proof. By induction on n.

Trivial when $n \leq 1$, so assume X is simply connected and $(n+1)$-truncated for $n > 0$.

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These are all simply connected, so by an earlier Theorem and Prop 2, we get another fibre sequence

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The fibre and base are $(n+1)$-truncated (using the Cor about EM spaces), and so $X_{(p)}$ is $(n+1)$-truncated as well. □

References

E. Rijke, M. Shulman, B. Spitters
Modalities in homotopy type theory, arXiv:1807.04155.

J.D. Christensen, M. Opie, E. Rijke and L. Scoccola.
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Thanks!

NO

YES

Maybe

Sort of...