Borsuk-Ulam in real-cohesive homotopy type theory

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Thanks to

- 2017 MRC in HoTT program
- Mike Shulman for his guidance and patience with three non-experts
- Univalence, which I’ll be recklessly using without mentioning I’m doing so
What’s this talk about?

Borsuk-Ulam is a result in classical algebraic topology. We want to import it into HoTT.
Outline of this talk

1. real-cohesive homotopy type theory
2. Borsuk-Ulam: algebraic topology vs. HoTT
3. proof sketch
real-cohesive homotopy type theory
Algebraic topology has many proofs that involve discontinuous constructions

For instance, Brouwer fixed-pt theorem

\[ \prod \sum f(x) = x \mid_{0} \]

No continuous way to pick \( x \)
discontinuous constructions

homotopy type theory
However, we don’t merely have HoTT... we have real-cohesive HoTT
real cohesive hott

discontinuous constructions
For spaces $X$ and $Y$, how can we make a **discontinuous** map

$$X \rightarrow Y$$

into a **continuous** map?

*Retopologize!*

$$\text{discrete}(X) \rightarrow Y \quad \text{or} \quad X \rightarrow \text{codiscrete}(Y)$$

There’s a ready-made theory for this... *Lawvere’s cohesive topoi*
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There’s a ready-made theory for this...*Lawvere’s cohesive topoi*
Discontinuity via cohesive topoi

\[ \flat X \to X \to \sharp X \]

Interpret \( \flat X \to Y \) or \( X \to \sharp Y \) as \textit{not necessarily continuous} maps from \( X \to Y \).
Interpret $♭X \to Y$ or $X \to ♯Y$ as *not necessarily continuous* maps from $X \to Y$. 

$♭X \to X \to ♯X$
Two concerns importing this to HoTT:

1) Algebraic topology trades in *spaces* not *higher inductive types*.
2) How can we retopologize when HoTT doesn’t have topologies (open sets)?
higher inductive types vs. spaces in HoTT
base : $S^1$
loop : base=base

$\{(x,y) \mid x^2 + y^2 = 1\}$
Lawvere’s theory of cohesive topoi has more to offer!

\[ \text{gives modality } \int : \text{SPACES} \to \text{SPACES} \]
- Cohesive topos: $\int$ is connected components
- HoTT: $\int$ is fundamental $\infty$-groupoid
- Cohesive topos: $\int$ is connected components
- HoTT: $\int$ is fundamental $\infty$-groupoid

\[
\int \vdash b \vdash \# \]

16
HoTT

+ \int 
\begin{align*}
\text{(suitably defined)}
\end{align*}
\text{cohesive homotopy type theory}

Notation

- \( S^1 := \{ (x, y) : x^2 + y^2 = 1 \} \)
- \( S^1 := \) higher inductive type

We want \( \int S^1 = S^1 \), but we’re not there yet.
HoTT
\[ + \int \vdash b \vdash \# \text{ (suitably defined)} \]
cohesive homotopy type theory

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incorporating topology into HoTT
The topology of a type $X$ is encoded in the type of “continuous paths” $\mathbb{R} \to X$.

Needed to define $\int$:
An axiom ensuring that $\int$ is constructed from continuous paths indexed by intervals in $\mathbb{R}$.

**Axiom $R^\flat$:**
A type $X$ is discrete iff $\text{const}: X \to (\mathbb{R} \to X)$ is an equivalence.
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**Axiom $R♭$:**
A type $X$ is *discrete* iff $\text{const}: X \to (\mathbb{R} \to X)$ is an equivalence.
real-cohesive homotopy type theory, a place where $\int S^1 = S^1$. 

**Axiom $R_b$:**

$X$ is discrete

iff

$\text{const} : X \cong (\mathbb{R} \to X)$
Borsuk-Ulam
Three related statements in classical algebraic topology:

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<td>There is no continuous map $f : S^n \to S^{n-1}$ with the property that there exists an $x \in S^n$ such that $f(-x) = -f(x)$.</td>
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Proof of BU-classic involves proving that

1. show $(\text{BU-classic}) \equiv (\text{BU-odd})$
2. show $\neg (\text{BU-odd}) \Rightarrow \neg (\text{BU-retract})$
3. hence $(\text{BU-retract}) \Rightarrow (\text{BU-classic})$.
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**BU-* in real-cohesive homotopy type theory:**

| BU-classic | \[
| \prod_{f : S^n \to \mathbb{R}^n} \sum_{x : S^n} f(-x) = f(x) |
| BU-odd | \[
| \prod_{f : S^n \to \mathbb{R}^n} \prod_{x : S^n} f(-x) = -f(x) \to \sum_{x : S^n} f(x) = 0 |
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Proof of BU-classic, strategy:

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2. show $\neg (\text{BU-odd}) \Rightarrow \neg (\text{BU-retract})$
3. hence $(\text{BU-retract}) \Rightarrow \neg \neg (\text{BU-odd})$
4. prove $(\text{BU-retract})$
5. conclude $\neg \neg (\text{BU-classic})$

$\neg \neg (\text{BU-classic}) \neq (\text{BU-classic})$ continuously

but

$\neg \neg (\text{BU-classic}) = (\text{BU-classic})$ discontinuously

Lemma: (Shulman) For $P$ a proposition, $\# P = \neg \neg P$
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It follows that

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\prod_{f : S^n \to \mathbb{R}^n} \sum_{x : S^n} f(-x) = f(x) = \# \prod_{f : S^n \to \mathbb{R}^n} \sum_{x : S^n} f(-x) = f(x)
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Hence (BU-retract) \(\Rightarrow\) \(\#\) (BU-classic).

Real-cohesive HoTT supports the **sharp Borsuk-Ulam theorem**.
It follows that

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Borsuk-Ulam

rc hott

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Sharp Borsuk-Ulam
To prove BU-retract

\[ \prod \sum f(-x) = -f(x) \to 0 \]

we will model the classical proof, which is

- Assume \( f : S^n \to S^{n-1} \) is odd and continuous
- Pass to orbits under \( \mathbb{Z}/2\mathbb{Z} \)-action: \( \hat{f} : \mathbb{R}P^n \to \mathbb{R}P^{n-1} \)
- This induces isomorphism on fundamental groups, \( \mathbb{Z}/2\mathbb{Z} \)
- Hurewicz theorem gives an isomorphism on \( H^1 \), hence we get a ring map \( \hat{f}^* : H^*(\mathbb{R}P^{n-1}, \mathbb{Z}/2\mathbb{Z}) \to H^*(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) \) such that
  \[ a : \mathbb{Z}/2\mathbb{Z}[a]/(a^{n-1}) \leftrightarrow b : \mathbb{Z}/2\mathbb{Z}[b]/(b^n) \]
- But then \( 0 = a^{n-1} \leftrightarrow b^{n-1} \neq 0 \). Contradiction.
To prove BU-retract

\[ \prod \sum f(-x) = -f(x) \rightarrow 0 \]

we will model the classical proof, which is

- Assume \( f : \mathbb{S}^n \rightarrow \mathbb{S}^{n-1} \) is odd and continuous
- Pass to orbits under \( \mathbb{Z}/2\mathbb{Z} \)-action: \( \hat{f} : \mathbb{RP}^n \rightarrow \mathbb{RP}^{n-1} \)
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- But then \( 0 = a^{n-1} \leftrightarrow b^{n-1} \neq 0 \). Contradiction.
Proof by contradiction are not permitted in intuitionistic logic

\[ \neg p, \Gamma \vdash p \]

\[ \Gamma \vdash \neg p \rightarrow 0 \]

\[ \Gamma \vdash \neg \neg p \]

This is actually a proof by negation, not contradiction

\[ p, \Gamma \vdash \neg p \]

\[ \Gamma \vdash \neg p \]

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\[ \Gamma \vdash \neg p \]
\[ \Gamma \vdash p \rightarrow 0 \]

which is allowed.
Four chunks of the real-cohesive HoTT proof:

- Define topological $S^n$
- Define topological $\mathbb{RP}^n$
- Define cohomology of $S^n$ and $\mathbb{RP}^n$ with $\mathbb{Z}/2\mathbb{Z}$-coefficients
- odd $f : S^n \to S^{n-1}$ induces contradiction (or, rather, negation).
Define $S^n$ topologically.

Per Shulman, $S^1$ is the coequalizer of

$$\text{id}, +1 : \mathbb{R} \to \mathbb{R}$$

giving $S^1 = \{(x, y) : x^2 + y^2 = 1\}$

Define higher dimensional spheres as pushouts:

\[
\begin{array}{ccc}
S^{n-1} \times \mathbb{R} & \overset{\text{fat } \partial}{\longrightarrow} & D^n \\
\downarrow \text{fat } \partial & & \downarrow \\
D^n & \longrightarrow & S^n
\end{array}
\]

Lemma: $S^n$ is a set.
Define $\mathbb{S}^n$ topologically.

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Define higher dimensional spheres as pushouts:

$$\mathbb{S}^{n-1} \times \mathbb{R} \xrightarrow{\text{fat } \partial} \mathbb{D}^n$$

**Lemma:** $\mathbb{S}^n$ is a set.
Define $\mathbb{R}P^n$ topologically using pushouts.

**Lemma:** The pushout of three sets over an injection is a set.

**Corollary:** $\mathbb{R}P^n$ is a set.
Define $\mathbb{R}P^n$ topologically using pushouts.

**Lemma:** The pushout of three sets over an injection is a set.

**Corollary:** $\mathbb{R}P^n$ is a set.
\[ H^n(X, R) := \|X \to K(R, n)\|_0 \]

Goals:

- Define a ring structure on \( H^* \) for \( S^n \) and \( \mathbb{RP}^n \)
- Compute \( H^* \) for \( S^n \) and \( \mathbb{RP}^n \)

Out strategy is inspired by Brunerie's doctoral thesis. Namely, work with EM-spaces \( K(R, n) \) then lift to cohomology.
Z/2Z-Cohomology for $S^n$ and $\mathbb{RP}^n$

For a type $X$ and ring $R$:

$$H^n(X, R) := \|X \to K(R, n)\|_0$$

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- Compute $H^*$ for $S^n$ and $\mathbb{RP}^n$

Out strategy is inspired by Brunerie’s doctoral thesis. Namely, work with EM-spaces $K(R, n)$ then lift to cohomology.
Define a *cup product* on EM-spaces:

\[
K(\mathbb{Z}/2\mathbb{Z}, n) \times K(\mathbb{Z}/2\mathbb{Z}, m) \xrightarrow{\sim} K(\mathbb{Z}/2\mathbb{Z}, n + m)
\]

\[
\pi \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\]

\[
K(\mathbb{Z}/2\mathbb{Z}, n) \otimes K(\mathbb{Z}/2\mathbb{Z}, m) \equiv K(\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}, n + m)
\]
Lift to $H^*$:

$$\sim: H^n(X, \mathbb{Z}/2\mathbb{Z}) \times H^m(X, \mathbb{Z}/2\mathbb{Z}) \to H^{n+m}(X, \mathbb{Z}/2\mathbb{Z})$$

$$\left( X \overset{\alpha}{\to} K(\mathbb{Z}/2\mathbb{Z}, n), X \overset{\beta}{\to} K(\mathbb{Z}/2\mathbb{Z}, m) \right)$$

is mapped to

$$X \overset{\langle \alpha, \beta \rangle}{\to} K(\mathbb{Z}/2\mathbb{Z}, n) \times K(\mathbb{Z}/2\mathbb{Z}, m) \overset{\sim}{\to} K(\mathbb{Z}/2\mathbb{Z}, n + m)$$

The remaining operations on $H^*(X, \mathbb{Z}/2\mathbb{Z})$ give a graded ring.
Lift to $H^*$:

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The remaining operations on $H^*(X, \mathbb{Z}/2\mathbb{Z})$ give a graded ring.
Use

- $K(\mathbb{Z}/2\mathbb{Z}, 0) := \mathbb{Z}/2\mathbb{Z}$
- $H^k(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) := ||\mathbb{R}P^n \to \mathbb{Z}/2\mathbb{Z}||_0$

to compute $H^0(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z})$

Use

- that $\mathbb{R}P^n$ is a pushout
- induction with Mayer-Vietoris

To compute $H^k(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z})$, for $k \geq 1$

(req's cohomology of $S^n$ and $D^n$ which are computed using MV and $D^n = 1$)
Use

- $K(\mathbb{Z}/2\mathbb{Z}, 0) := \mathbb{Z}/2\mathbb{Z}$
- $H^k(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) := ||\mathbb{RP}^n \to \mathbb{Z}/2\mathbb{Z}||_0$

to compute $H^0(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z})$

Use

- that $\mathbb{RP}^n$ is a pushout
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to compute $H^k(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z})$, for $k \geq 1$

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Use

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Use

- That $\mathbb{R}P^n$ is a pushout
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(req's cohomology of $\mathbb{S}^n$ and $\mathbb{D}^n$ which are computed using MV and $\mathbb{D}^n = 1$)
The results are in:

\[ H^k(S^n, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & k = 0, n; \\ 0, & \text{else} \end{cases} \]

\[ H^k(D^n, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & k = 0; \\ 0, & \text{else} \end{cases} \]

\[ H^k(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & k = 2, 3, \ldots, n; \\ 0, & k \geq n + 1 \end{cases} \]

*(note* \( n \geq 2 \))

In particular:

\[ H^*(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x]/(x^{n+1}) \]
The results are in:

\[
H^k(S^n, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} 
\mathbb{Z}/2\mathbb{Z}, & k = 0, n; \\
0, & \text{else}
\end{cases}
\]

\[
H^k(D^n, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} 
\mathbb{Z}/2\mathbb{Z}, & k = 0; \\
0, & \text{else}
\end{cases}
\]

\[
H^k(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} 
\mathbb{Z}/2\mathbb{Z}, & k = 2, 3, \ldots, n; \\
0, & k \geq n + 1
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(note \( n \geq 2 \))

In particular:

\[
H^*(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x]/(x^{n+1})
\]
Prove BU-retract

Recall,

- \( f : \mathbb{S}^n \to \mathbb{S}^{n-1} \) is continuous and odd
- \( \hat{f} : \mathbb{RP}^n \to \mathbb{RP}^{n-1} \) is the induced map

Apply \( H^1(\cdot, \mathbb{Z}/2\mathbb{Z}) \) to \( \hat{f} \) to get

\[
\hat{f}^* : H^1(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{RP}^{n-1}, \mathbb{Z}/2\mathbb{Z})
\]

More concretely

\[
\hat{f}^* : \| \mathbb{RP}^n \to \mathbb{RP}^2 \| \to \| \mathbb{RP}^{n-1} \to \mathbb{RP}^2 \|
\]

\[\alpha \mapsto \hat{f}\alpha\]

Note: \( \alpha \) non-trivial implies \( \hat{f}\alpha \) non-trivial.
Prove BU-retract

Recall,

- $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$ is continuous and odd
- $\hat{f} : \mathbb{RP}^n \to \mathbb{RP}^{n-1}$ is the induced map

Apply $H^1(-, \mathbb{Z}/2\mathbb{Z})$ to $\hat{f}$ to get

$$\hat{f}^* : H^1(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{RP}^{n-1}, \mathbb{Z}/2\mathbb{Z})$$

More concretely

$$\hat{f}^* : \|\mathbb{RP}^n \to \mathbb{RP}^2\| \to \|\mathbb{RP}^{n-1} \to \mathbb{RP}^2\|$$

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Recall, 

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More concretely

\[
\hat{f}^* : \|\mathbb{R}P^n \rightarrow \mathbb{R}P^2\| \rightarrow \|\mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^2\|
\]

\[
\alpha \mapsto \hat{f}\alpha
\]

**Note:** \( \alpha \) non-trivial implies \( \hat{f}\alpha \) non-trivial.
The generator of

\[ H^*(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x]/(x^{n+1}) \]

live in \( H^1 \).

If follows: \( f : S^n \to S^{n-1} \) induces a map on cohomology

\[ \mathbb{Z}/2\mathbb{Z}[x]/(x^{n-1}) \to \mathbb{Z}/2\mathbb{Z}[y]/(y^n) \]

preserving the generator: \( x \mapsto y \)

But then \( 0 = x^{n-1} \mapsto y^{n-1} \neq 0 \).

Contradiction (or rather, negation).
The generator of

\[ H^\ast(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x]/(x^{n+1}) \]

live in \( H^1 \).

If follows: \( f : \mathbb{S}^n \to \mathbb{S}^{n-1} \) induces a map on cohomology

\[ \mathbb{Z}/2\mathbb{Z}[x]/(x^{n-1}) \to \mathbb{Z}/2\mathbb{Z}[y]/(y^n) \]

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\[ \mathbb{Z}/2\mathbb{Z}[x]/(x^{n-1}) \to \mathbb{Z}/2\mathbb{Z}[y]/(y^n) \]

preserving the generator: \( x \mapsto y \)

But then \( 0 = x^{n-1} \mapsto y^{n-1} \neq 0 \).

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The generator of

\[ H^* (\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x]/(x^{n+1}) \]

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We have proved BU-retract, hence sharp Borsuk-Ulam as desired.

Thank you.
We have proved BU-retract, hence sharp Borsuk-Ulam as desired.

Thank you.