Borsuk-Ulam in real-cohesive homotopy type theory

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Thanks to

- 2017 MRC in HoTT program
- Mike Shulman for his guidance and patience with three **non**-experts
- Univalence, which I'll be recklessly using without mentioning I'm doing so

What's this talk about?

Borsuk-Ulam is a result in **classical algebraic topology**. We want to import it into HoTT.

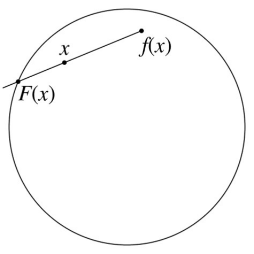
Outline of this talk

1. real-cohesive homotopy type theory
2. Borsuk-Ulam: algebraic topology vs. HoTT
3. proof sketch

real-cohesive homotopy type theory

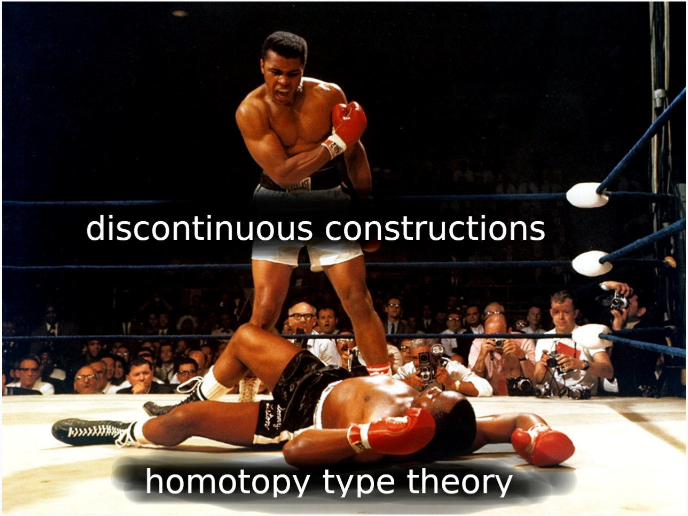
Algebraic topology has many proofs that involve discontinuous constructions

For instance, Brouwer fixed-pt theorem



$$\left\| \prod_{f\,:\, D^2 \to D^2} \sum_{x : D^2} f(x) = x \right\|_0$$

No continuous way to pick x


discontinuous constructions
homotopy type theory

However, we don't merely have HoTT...

we have real-cohesive HoTT


real cohesive hott
discontinuous constructions

For spaces X and Y, how can we make a **discontinuous** map

$$X \to Y$$

into a **continuous** map?

Retopologize!

$$\mathrm{discrete}(X) \to Y \qquad \text{or} \qquad X \to \mathrm{codiscrete}(Y)$$

There's a ready-made theory for this... *Lawvere's cohesive topoi*

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Discontinuity via cohesive topoi

$$
\begin{tikzcd}
\text{SPACES} \arrow[d, shift left=8, "\text{forget}"] \\
\text{POINTS} \arrow[u, shift left=16, "\text{discrete}"] \arrow[u, shift right=16, "\text{codiscrete}"'] \arrow[u, phantom, shift left=12, "\dashv"] \arrow[u, phantom, shift right=4, "\dashv"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\text{SPACES} \arrow[d, shift right=10, "\sharp := \text{codiscretize}"'] \\
\text{SPACES} \arrow[u, shift left=10, "\flat := \text{discretize}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
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$$\flat X \to X \to \sharp X$$

Interpret $\flat X \to Y$ or $X \to \sharp Y$ as *not necessarily continuous* maps from $X \to Y$.

Discontinuity via cohesive topoi

$$\begin{tikzcd}[row sep=huge]
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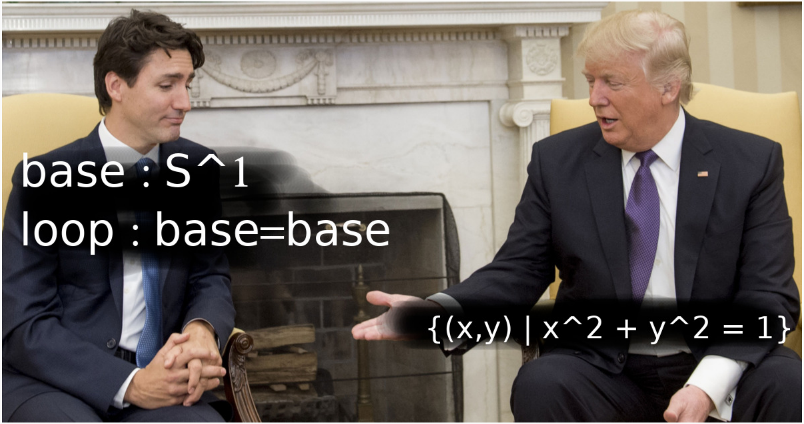
Interpret $\flat X \to Y$ or $X \to \sharp Y$ as *not necessarily continuous* maps from $X \to Y$.

Two concerns importing this to HoTT:

1) Algebraic topology trades in *spaces* not *higher inductive types*.
2) How can we retopologize when HoTT doesn't have topologies (open sets)?

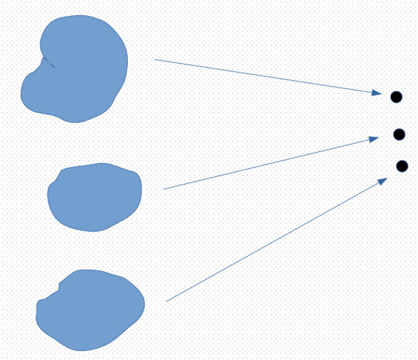
higher inductive types vs. spaces in HoTT


base : S^1
loop : base=base
{(x,y) | x^2 + y^2 = 1}

Lawvere's theory of cohesive topoi has more to offer!

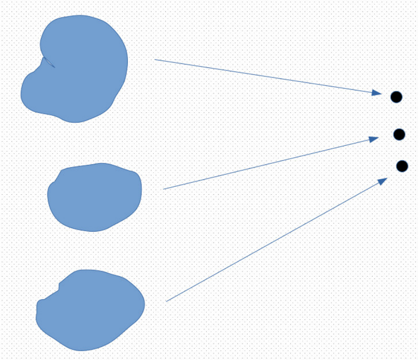
$$\begin{tikzcd}[row sep=huge, column sep=large]
\text{SPACES} \arrow[d, shift right=12, "\text{connected components}"'] \arrow[d, shift right=8, phantom, "\dashv"] \arrow[d, shift right=0, phantom, "\dashv"] \arrow[d, shift left=4, "\text{forget}"] \arrow[d, shift left=8, phantom, "\dashv"] \\
\text{POINTS} \arrow[u, shift right=4, "\text{discrete}"'] \arrow[u, shift left=12, "\text{codiscrete}"']
\end{tikzcd}$$

gives modality $\int : \text{SPACES} \to \text{SPACES}$



- Cohesive topos: $\int$ is connected components
- HoTT: $\int$ is fundamental ∞-groupoid

$$\int \dashv \flat \dashv \sharp$$



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$$\begin{array}{rl} & \text{HoTT} \\ + & \int \dashv \flat \dashv \sharp \text{ (suitably defined)} \\ \hline & \text{cohesive homotopy type theory} \end{array}$$

Notation

- $\mathbb{S}^1 :\equiv \{(x, y) : x^2 + y^2 = 1\}$
- $\mathrm{S}^1 :\equiv$ higher inductive type

We want $\int \mathbb{S}^1 = \mathrm{S}^1$, but we're not there yet.

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incorporating topology into HoTT

The topology of a type $\mathtt{X}$ is encoded in the type of "continuous paths" $\mathbb{R} \to \mathtt{X}$.

Needed to define $\int$:
An axiom ensuring that $\int$ is constructed from continuous paths indexed by intervals in $\mathbb{R}$.

Axiom $R\flat$**:**
A type $\mathtt{X}$ is *discrete* iff $\mathtt{const}\colon \mathtt{X} \to (\mathbb{R} \to \mathtt{X})$ is an equivalence.

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\mathrm{TYPES} \arrow[d, shift right=9, "\infty\text{-groupoid}"'] \arrow[d, phantom, shift right=6, "\dashv"] \arrow[d, phantom, "\dashv"] \arrow[d, phantom, shift left=6, "\dashv"] \arrow[d, shift left=3, "\mathrm{forget}"] \arrow[from=d, shift right=3, "\mathrm{discrete}"'] \arrow[from=d, shift left=9, "\mathrm{codiscrete}"'] \\
\mathrm{TYPES}
\end{tikzcd}$$

$+$

Axiom $R\flat$**:**

$\mathtt{X}$ is *discrete*

iff

$\mathtt{const}\colon \mathtt{X} \xrightarrow{=} (\mathbb{R} \to \mathtt{X})$

—equals—

real-cohesive homotopy type theory, a place where $\int \mathbb{S}^1 = \mathtt{S}^1$.

Borsuk-Ulam

Three related statements in classical algebraic topology:

BU-classic	For any continuous map $f\colon S^n \to \mathbb{R}^n$, there exists an $x \in S^n$ such that $f(x) = f(-x)$.
BU-odd	For any continuous map $f\colon S^n \to \mathbb{R}^n$ with the property that $f(-x) = -f(x)$, there is an $x \in S^n$ such that $f(x) = 0$
BU-retract	There is no continuous map $f\colon S^n \to S^{n-1}$ with the property that there exists an $x \in S^n$ such that $f(-x) = -f(x)$.

Proof of BU-classic involves proving that

1. show (BU-classic) $\simeq$ (BU-odd)
2. show $\neg$ (BU-odd) $\Rightarrow$ $\neg$ (BU-retract)
3. hence (BU-retract) $\Rightarrow$ (BU-classic).
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BU-* in real-cohesive homotopy type theory:

BU-classic	$\left\Vert \prod_{f : \mathbb{S}^n \to \mathbb{R}^n} \sum_{x : \mathbb{S}^n} f(-x) = f(x) \right\Vert$
BU-odd	$\left\Vert \prod_{f : \mathbb{S}^n \to \mathbb{R}^n} \prod_{x : \mathbb{S}^n} f(-x) = -f(x) \to \sum_{x : \mathbb{S}^n} f(x) = 0 \right\Vert$
BU-retract	$\left\Vert \prod_{f : \mathbb{S}^n \to \mathbb{S}^{n-1}} \sum_{x : \mathbb{S}^n} f(-x) = -f(x) \to 0 \right\Vert$

Proof of BU-classic, strategy:

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$\neg\neg$ (BU-classic) $\neq$ (BU-classic) continuously

but

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Lemma: (Shulman) For P a proposition, $\sharp P = \neg\neg P$

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Lemma: (Shulman) For P a proposition, $\sharp P = \neg\neg P$

It follows that

$$\neg\neg\left\Vert \prod_{f\,:\,\mathbb{S}^n \to \mathbb{R}^n} \sum_{x\,:\,\mathbb{S}^n} f(-x) = f(x) \right\Vert = \sharp \left\Vert \prod_{f\,:\,\mathbb{S}^n \to \mathbb{R}^n} \sum_{x\,:\,\mathbb{S}^n} f(-x) = f(x) \right\Vert$$

Hence (BU-retract) $\Rightarrow \sharp$ (BU-classic).

Real-cohesive HoTT supports the **sharp Borsuk-Ulam theorem**.

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Real-cohesive HoTT supports the **sharp Borsuk-Ulam theorem**.

Borsuk-Ulam
rc hott
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Sharp Borsuk-Ulam

To prove BU-retract

$$\left\| \prod_{f : \mathbb{S}^n \to \mathbb{S}^{n-1}} \sum_{x : \mathbb{S}^n} f(-x) = -f(x) \to 0 \right\|$$

we will model the classical proof, which is

- Assume $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$ is odd and continuous
- Pass to orbits under $\mathbb{Z}/2\mathbb{Z}$-action: $\hat{f} : \mathbb{RP}^n \to \mathbb{RP}^{n-1}$
- This induces isomorphism on fundamental groups, $\mathbb{Z}/2\mathbb{Z}$
- Hurewicz theorem gives an isomorphism on H^1, hence we get a ring map $\hat{f}^* : H^*(\mathbb{RP}^{n-1}, \mathbb{Z}/2\mathbb{Z}) \to H^*(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z})$ such that

$$a : \mathbb{Z}/2\mathbb{Z}[a]/(a^{n-1}) \mapsto b : \mathbb{Z}/2\mathbb{Z}[b]/(b^n)$$

- But then $0 = a^{n-1} \mapsto b^{n-1} \neq 0$. Contradiction.

To prove BU-retract

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Proof by contradiction are not permitted in intuitionistic logic

$$\cfrac{\cfrac{\neg p, \Gamma \vdash p}{\Gamma \vdash \neg p \to 0}}{\Gamma \vdash \neg\neg p}$$

This is actually a proof by negation, not contradiction

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Four chunks of the real-cohesive HoTT proof:

- Define topological $\mathbb{S}^n$
- Define topological $\mathbb{RP}^n$
- Define cohomology of $\mathbb{S}^n$ and $\mathbb{RP}^n$ with $\mathbb{Z}/2\mathbb{Z}$-coefficients
- odd $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$ induces contradiction (or, rather, negation).

Define $\mathbb{S}^n$ topologically.

Per Shulman, $\mathbb{S}^1$ is the coequalizer of

$$\mathrm{id}, +1 \colon \mathbb{R} \to \mathbb{R}$$

giving $\mathbb{S}^1 = \{(x, y) : x^2 + y^2 = 1\}$

Define higher dimensional spheres as pushouts:

$$\begin{tikzcd}
\mathbb{S}^{n-1} \times \mathbb{R} \arrow[r, "\text{fat } \partial"] \arrow[d, "\text{fat } \partial"'] & \mathbb{D}^n \arrow[d] \\
\mathbb{D}^n \arrow[r] & \mathbb{S}^n \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Lemma: $\mathbb{S}^n$ is a set.

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Lemma: $\mathbb{S}^n$ is a set.

Define $\mathbb{RP}^n$ topologically using pushouts.

$$\begin{tikzcd} \mathbb{S}^1 \times \mathbb{R} \arrow[r, "\text{fat } 2\partial"] \arrow[d, "\text{fat } \partial"'] & \mathbb{D}^2 \arrow[d] \\ M \arrow[r] & \mathbb{RP}^2 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S}^n \times \mathbb{R} \arrow[r, "\partial"] \arrow[d, "\text{fat } \partial"'] & \mathbb{D}^{n+1} \arrow[d] \\ \mathbb{RP}^n \arrow[r] & \mathbb{RP}^{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Lemma: The pushout of three sets over an injection is a set.

Corollary: $\mathbb{RP}^n$ is a set.

Define $\mathbb{RP}^n$ topologically using pushouts.

$$
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\mathbb{S}^1 \times \mathbb{R} \arrow[r, "\text{fat } 2\partial"] \arrow[d, "\text{fat } \partial"'] & \mathbb{D}^2 \arrow[d] \\
M \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathbb{RP}^2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S}^n \times \mathbb{R} \arrow[r, "\partial"] \arrow[d, "\text{fat } \partial"'] & \mathbb{D}^{n+1} \arrow[d] \\
\mathbb{RP}^n \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathbb{RP}^{n+1}
\end{tikzcd}
$$

Lemma: The pushout of three sets over an injection is a set.

Corollary: $\mathbb{RP}^n$ is a set.

$\mathbb{Z}/2\mathbb{Z}$-Cohomology for $\mathbb{S}^n$ and $\mathbb{RP}^n$

For a type X and ring R:

$$H^n(X, R) :\equiv \|X \to K(R, n)\|_0$$

Goals:

- Define a ring structure on H^* for $\mathbb{S}^n$ and $\mathbb{RP}^n$
- Compute H^* for $\mathbb{S}^n$ and $\mathbb{RP}^n$

Out strategy is inspired by Brunerie's doctoral thesis. Namely, work with EM-spaces $K(R, n)$ then lift to cohomology.

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Define a *cup product* on EM-spaces:

$$\begin{tikzcd}
K(\mathbb{Z}/2\mathbb{Z}, n) \times K(\mathbb{Z}/2\mathbb{Z}, m) \arrow[rr, "\smile"] \arrow[d, "\pi"'] & & K(\mathbb{Z}/2\mathbb{Z}, n+m) \\
K(\mathbb{Z}/2\mathbb{Z}, n) \wedge K(\mathbb{Z}/2\mathbb{Z}, m) \arrow[d, "||-||_{n+m}"'] & & \\
||K(\mathbb{Z}/2\mathbb{Z}, n) \wedge K(\mathbb{Z}/2\mathbb{Z}, m)||_{n+m} \arrow[rr, "="'] & & K(\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}, n+m) \arrow[uu, "="']
\end{tikzcd}$$

Lift to H^*:

$$\smile \colon H^n(X, \mathbb{Z}/2\mathbb{Z}) \times H^m(X, \mathbb{Z}/2\mathbb{Z}) \to H^{n+m}(X, \mathbb{Z}/2\mathbb{Z})$$

$$\left(X \xrightarrow{\alpha} K(\mathbb{Z}/2\mathbb{Z}, n), X \xrightarrow{\beta} K(\mathbb{Z}/2\mathbb{Z}, m)\right)$$

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The results are in:

$$H^k(\mathbb{S}^n, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & k = 0, n; \\ 0, & \text{else} \end{cases}$$

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Prove BU-retract

Recall,

- $f\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$ is continuous and odd
- $\hat{f}\colon \mathbb{RP}^n \to \mathbb{RP}^{n-1}$ is the induced map

Apply $H^1(-, \mathbb{Z}/2\mathbb{Z})$ to $\hat{f}$ to get

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More concretely

$$\begin{aligned} \hat{f}^*\colon \left\|\mathbb{RP}^n \to \mathbb{RP}^2\right\| &\to \left\|\mathbb{RP}^{n-1} \to \mathbb{RP}^2\right\| \\ \alpha &\mapsto \hat{f}\alpha \end{aligned}$$

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But then $0 = x^{n-1} \mapsto y^{n-1} \neq 0$.

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