Constructive Presheaf Models of Univalence

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Acknowledgments

Parts are joint work with Steve Awodey and Emily Riehl

Parts are also joint with Evan Cavallo and Christian Sattler

The last part comes from further discussion with Christian Sattler

Type theoretic construction of Model Structures

As explained in Steve's talk, we reverse the direction

QMS on Simplicial Sets $\rightarrow$ model of univalent type theory

to

models of univalent type theory $\rightarrow$ QMS on presheaf categories

One key point is to have a *fibrant* universe (of fibrant types)

Type theoretic construction of Model Structures

Furthermore these QMS satisfy

-Frobenius (and right properness)

-Equivalence Extension Property

-Fibration Extension Property

Type theoretic construction of Model Structures, some features

-These models can be developed in a *constructive* meta theory

-They can be developed using the internal language of presheaf categories (model of dependent type theory), and they have been *formalised* (in Agda)

–Condition for fibrant objects: same as in Cisinski's work, lifting property w.r.t. "generalized open box", i.e. push-out product of cofibration and end point inclusion in the interval

-Needs (without connections) to be generalized to: *generic* point inclusion

-The interval has to be *tiny* (Δ^1 is *not* tiny)

Cartesian Cubical Sets

In particular, this works for *cartesian* cubes (cf. Steve's talk)

A model of univalent type theory is presented (and Agda formalised) in

Cartesian Cubical Type Theory, ABCFHL

Cartesian Cubical Sets

Cartesian cubes are interesting classically, since the base category $\square$ is *generalized Reedy* (cf. Emily's talk)

We get a QMS on cartesian cubes

We say that a presheaf F (non necessarily fibrant) is weakly contractible if the canonical map $F \to 1$ is an equivalence

Christian Sattler found out that the quotient of a square by swapping is *not* weakly contractible for this QMS

As explained in Emily's talk, this issue is solved by imposing the further property of *equivariance*

Cartesian Cubical Sets

The main facts (in particular the ones that imply that the universe of fibrant types is fibrant) have been checked formally in Agda (Evan Cavallo)

For this QMS, all quotients $\mathbb{I}^n/G$, where G finite group, are weakly contractible

Cartesian Cubical Sets and Generalized Reedy Property

We can *classically* build a section (Excluded-Middle) of

$X + \neg X \qquad (X : U,\ h : \text{isProp}\ X)$

building it by *induction on dimension*

In particular, in this model Bool is classically equivalent to $\text{hProp}(U)$!

Cartesian Cubical Sets and Generalized Reedy Property

Classically one can also prove

The triangulation map cSet → sSet *is a Quillen equivalence* (Christian Sattler)

(pre)Sheaf models

For building a QMS, we need the structure $(\mathcal{E}, \Phi, \mathbb{I}, V)$, where $\mathbb{I}$ is *tiny*

In particular, we can build in this way a QMS on *cubical presheaves*, i.e. presheaves over $\mathcal{C} \times \Box$ where $\mathcal{C}$ is any small category

We define a new interval $\tilde{\mathbb{I}}(X, J) = \mathbb{I}(J)$ which is still tiny

(pre)Sheaf models

Let F be a(n ordinary) presheaf over $\mathcal{C}$ and S a sieve on an object X of $\mathcal{C}$

A collection of compatible local data (for the usual sheaf condition) for S is a collection of elements $a(f)$ in $F(Y)$ for $f : Y \to X$ s.t. $a(fg) = a(f)g$

This is the (ordinary) limit of the diagram $(Y, f) \longmapsto F(Y)$ over S

This defines a new presheaf $\mathsf{LD}_S(F)$ with a canonical map $\eta_S : F \to \mathsf{LD}_S(F)$

F is a sheaf if η_S is an isomorphism for each covering sieve S

Stacks

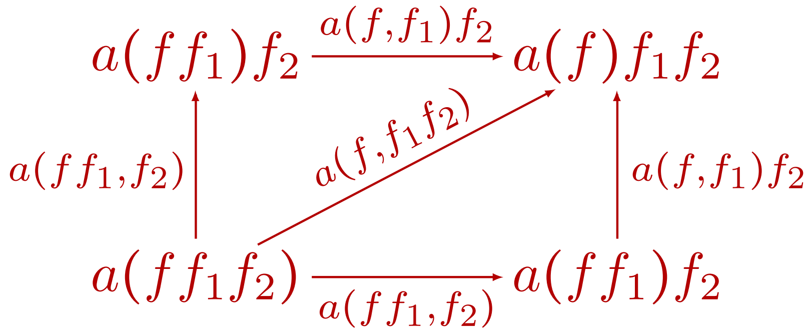
For a *cubical* presheaf F, a collection of compatible local data can be seen instead as a *homotopy* limit over S of the diagram $(Y,f) \longmapsto F(Y)$

This defines a new (cubical) presheaf $\mathsf{LD}_S(F)$

Intuitively, we have a path $a(fg) \to a(f)g$, so $a(fg)$ is not strictly equal to $a(f)g$, for $f : Y \to X \ , g : Z \to Y$

In general, a m-cube in $\mathsf{LD}(F)(X)(\mathbb{I}^m)$ is given by a collection of *symmetric* n-cubes $a(f, f_1, \dots, f_n)$ in $F(X_{n+1}, \mathbb{I}^{m+n})$ for $f : X_1 \to X$ in S and $f_i : X_{i+1} \to X_i$

Compatible local data



Symmetric square $a(f, f_1, f_2)$

(pre)Sheaf models

As in the sheaf case, there is a canonical map $\eta_S : F \to \mathsf{LD}_S(F)$

F is a *stack* if η_S is an equivalence for each S covering sieve

For this, it is enough to have a patch function $p_S : \mathsf{LD}_S(F) \to F$

Patch function: $p_S \eta_S$ is path equal to the identity

(pre)Sheaf models

Special case: S total sieve

For usual presheaf of *sets* the map η_S is an isomorphism in this case

We have a patch function

$$\begin{aligned} \mathsf{LD}_S(F)(X) &\longrightarrow F(X) \\ a &\longmapsto a(1_X) \end{aligned}$$

For stacks, in general we don't have a *functorial* patch function!

(pre)Sheaf models

$\mathsf{cobar}(F) = \mathsf{LD}_S(F)$ for the *total sieve* S

$\mathsf{cobar}(F)(X)$ is homotopy limit over $\mathcal{C}/X$ of the diagram $(Y, f) \mapsto F(Y)$

This defines a *left exact modality*

Hence a model of univalent type theory, from which we can build a new QMS

New model defined by cobar

In this new model, a fibrant type is a presheaf F such that the canonical map

$\eta : F \to \mathsf{cobar}(F)$

has a patch function

$p : \mathsf{cobar}(F) \to F$

i.e. a map p such that $p\eta$ is path equal to the identity on F

We expect to have

The QMS constructed to this model of type theory is the injective *Quillen model structure*

(pre)Sheaf models

At least we can check that, for this "localised" QMS, a cofibration which is pointwise a trivial cofibration is a trivial cofibration

This follows Mike Shulman's insight in the paper

All $(\infty, 1)$-toposes have strict univalent universes

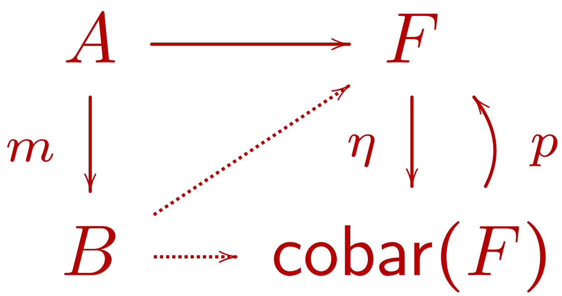
(pre)Sheaf models

If we have a cofibration $m : A \to B$ *which is* pointwise *a trivial cofibration and* F *is fibrant, any map* $A \to F$ *extends to a map* $B \to \mathsf{cobar}(F)$

$$\begin{array}{ccc} A & \longrightarrow & F \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \eta} \\ B & \dashrightarrow & \mathsf{cobar}(F) \end{array}$$

(pre)Sheaf models

Hence if $F \to \mathrm{cobar}(F)$ has a patch function $p : \mathrm{cobar}(F) \to F$, we can extend the map $A \to F$ to a map $B \to F$ first up to homotopy and then strictly since m is a cofibration



Example 1

Cubical presheaves over the poset $0 \leqslant 1$

In this case $\mathsf{cobar}(F)$ can be seen as an exponential F^C for some C

Any presheaf is already modal: we don't need to localize

A presheaf is exactly a fibration $F_1 \to F_0$ of cubical sets

Example 2

Cubical presheaves over the poset $X_0 \geqslant X_1 \geqslant X_2 \geqslant \dots$

In this case, we need to localise

Example 3

Model of *parametrised pointed types*

Cubical presheaves over category: X with an idempotent endomap f