Toward a Homotopy Tripos for Higher Realizability

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Main Goal

Our simple goal is to discuss the following definition:

Def. Higher Turing category

A *higher Turing category* is a cartesian restriction ∞-category $\mathcal{C}$ with an object A of $\mathcal{C}$ and coherent application $\bullet : A \times A \to A$, such that every object X of $\mathcal{C}$ is a homotopy retract of A.

and its relevance to computable interpretations of univalence in HoTT.

Categorifying Recursion Theory

- The notion of *function partiality* is foreign to type theories, both natively and as the internal logics of categories.
- Models of computability as found in recursion theory reflect this fact, e.g. PCAs are models of *untyped* lambda calculi.

Categorical folks interested in computability have therefore introduced many ways to apply intrinsically categorical methods to recursion theory:

- Partial map categories (Longo, Moggi, Robinson, Rosolini...)
- Dominical/recursion categories (Di Paola, Heller, Montagna, Lengyel...)
- The recursive topos (Mulry...)
- Arithmetical universes (Joyal...)
- Realizability toposes (Hyland, Pitts, Johnstone...)
- The effective topos (Hyland...)
- Restriction categories (Cockett, Lack...)

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The minimalism and equationality of restriction categories make them our starting point for "homotopifying" recursion theory with a view towards a realizable interpretation of univalence.

Categorifying Recursion Theory

Def. Restriction category

A **restriction category** $(\mathcal{C}, ^{-})$ is a category $\mathcal{C}$ along with a combinator $^{-} : \mathrm{Arr}\mathcal{C} \to \mathrm{Arr}\mathcal{C}$ assigning to each arrow $f : A \to B$ in $\mathcal{C}$ an $\bar{f} : A \to A$ such that:

i) $f\bar{f} = f$
ii) for all $g : \mathrm{Hom}_{\mathcal{C}}(A, C)$, $\bar{g}\bar{f} = \bar{f}\bar{g}$
iii) for all $g : \mathrm{Hom}_{\mathcal{C}}(A, C)$, $\overline{g\bar{f}} = \bar{g}\bar{f}$
iv) for all $g : \mathrm{Hom}_{\mathcal{C}}(B, C)$, $\bar{g}f = f\overline{gf}$

NB - A morphism $f : A \to B$ in $\mathcal{C}$ is called **total** if $\bar{f} = id_A$.

- Functors (properly, restriction functors) of restriction categories preserve the partiality structures.
- Objects and total maps of $\mathcal{C}$ form a subcategory $\mathrm{Tot}(C, ^{-}) \hookrightarrow (\mathcal{C}, ^{-})$ in this sense.
- Examples: **ParSet**, **Rec**, **ParTop**...

Restriction Categories - Some Properties

Diagrams in a restriction category do not commute equationally, but as inequalities in the poset order induced by restriction. E.g.

- a **restricted final object 1** for a restriction category $(\mathcal{C}, {}^{-})$ has the universal property:

$$
\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, dashed, "\exists!_A"] & \\
B \arrow[r, dashed, "\exists!_B"'] \arrow[ur, phantom, "\leqslant" near start] & {}
\end{tikzcd}
$$

- and **restricted binary products** satisfy:

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\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[d, dashed, "\exists! d"] \arrow[dr, "g"] & \\
B \arrow[ur, phantom, "\geqslant" near end] & B \times C \arrow[l, "\pi_B"] \arrow[r, "\pi_C"'] & C \arrow[ul, phantom, "\leqslant" near end]
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where the projections are total.

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Def.

A restriction category with restricted binary products and a restricted final object is called a **cartesian restriction category**.

Turing (1-)Categories

Def. Turing Category

A **Turing category** is a cartesian restriction category $(\mathcal{C}, ^{-})$ with a fixed object A and morphism $\bullet : A \times A \to A$ having the following universal property: for each $\mathcal{C}$-morphism $f : X \to Y$ there is a section $s : Y \to A$ and retract $r : A \to X$, along with a total map $h : \mathbf{1} \to A$ satisfying the diagram:

$$
\begin{tikzcd}
A \times A \arrow[r, "\bullet"] & A \arrow[r, "r"] & X \arrow[ddll, "f"] \\
A \times 1 \simeq A \arrow[u, "\mathsf{id_A} \times h"] \arrow[ur, "sfr"'] & & \\
Y \arrow[u, "s"] & &
\end{tikzcd}
$$

That is, each map in $\mathcal{C}$ ia A-computable up to sections and retractions. Note that this commutes equationally; the products shown are restricted products.

Turing (1-)categories

The universal object in $\mathcal{C}$, the Turing object, should be thought of as Gödel-encoding all maps $f : X \to Y$ in $\mathcal{C}$ via its application $A \times X \to Y$, represented by the global section $h : 1 \to A$.

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Basic examples:

- **Rec**, of natural numbers $n, m \in \mathbf{N}$ and partial recursive functions $\mathbf{N}^n \to \mathbf{N}^m$, with universal applicative structure:

$$\begin{tikzcd} \mathbf{N} \times 1 \simeq \mathbf{N} \arrow[d, dashed, "\exists \text{ total } h \,:\, 1\times h"'] \arrow[dr, "f_i"] & \\ \mathbf{N} \times \mathbf{N} \arrow[r, "\langle - - \rangle"'] & \mathbf{N} \end{tikzcd}$$

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  where the representation is $\langle i, n \rangle = f_i(n)$, the ith computable function.
- Conversely, any PCA gives rise to a Turing category, via its computable map category.
- The Karoubi envelope (idempotent splitting) of any Turing category is a Turing category.

Turing Objects as Relative PCAs

Just as the computable map category of a PCA forms a Turing category, a Turing object A of $\mathcal{C}$ and its Turing morphism $\bullet : A \times A \to A$ forms a PCA in $\mathcal{C}$.

Def.

A (relative) PCA A is a combinatory complete partial applicative system in a cartesian restriction category $\mathcal{D}$.

- *Partial applicative system* := a morphism $\bullet : A \times A \to A$ in $\mathcal{D}$
- *Completeness in* $\mathcal{D}$:= finite powers A^n and A-computable morphisms form a well-defined cartesian restriction subcategory of $\mathcal{D}$

Caveat. Not every PCA in a Turing category $\mathcal{C}$ is a Turing object for $\mathcal{C}$.

Note on Formalization

The preceding material (and more), but not the material to follow, has been formalized in Coq by Vinogradova, Felty, and Scott (2018), code available at:

github.com/polinavino/Turing-Category-Formalization

Realizability Toposes from Turing Categories

The Turing object-to-realizability topos construction works much as in the case for classical PCAs, but is again a purely categorical formulation. Let $\mathcal{F} : \mathcal{D} \to \mathcal{C}$ be a restriction functor.

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- Let β be an $\mathcal{D}$-assembly with object $\mathcal{F}(B) \times Y \in C$. Then a **morphism of assemblies** $f : \alpha \to \beta$ is a map $f : X \to Y$ in $\mathcal{C}$ s.t. there exists a $\mathcal{D}$-morphism $d : A \to B$ and:

  i) $(\mathcal{F}(d) \times f) \circ \alpha = \beta \circ (\mathcal{F}(d) \times f) \circ \alpha$,

  ii) $\overline{(\mathsf{id}_{\mathcal{F}(A)} \times f) \circ \alpha} = \overline{(\mathcal{F}(d) \times f) \circ \alpha}$.

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- These assemblies and their morphisms form a (restriction) category $ASM(\mathcal{F})$
- There is now a forgetful functor $\partial : ASM(\mathcal{F}) \longrightarrow \mathcal{C}$.

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When $\mathcal{C}$ and $\mathcal{D}$ are Turing categories, ∂ is a fibration, and this fibration is a tripos. In fact, $\partial(ASM(\mathcal{F}))$ is a *realizability tripos* whose internal language defines a realizability topos.

Realizability Toposes: An Extensional Characterization

"Frey's axioms" (2014/2018): a resolution of Johnstone's complaint (c. 2010)?

Theorem (Frey 2014)

A (locally small) category $\mathcal{C}$ is a realizability topos iff:

I) $\mathcal{C}$ is exact and locally cartesian closed,

II) $\mathcal{C}$ has enough projectives and the full subcategory $\mathrm{Proj}(\mathcal{C}) \hookrightarrow \mathcal{C}$ has all finite limits,

III) The global section functor

$$\Gamma := \mathcal{C}(\bullet, -) : \mathcal{C} \longrightarrow \mathbf{Set}$$

has right adjoint $\nabla : \mathbf{Set} \hookrightarrow \mathcal{C}$, a reflective inclusion s.t. $\nabla\Gamma$ preserves finite limits and the idempotent closure operator. The modality ∇ factors through $\mathrm{Proj}(\mathcal{C})$.

IV) Finally, there is a $\nabla\Gamma$-separated object $D \in \mathrm{Proj}(\mathcal{C})$ s.t. regular epics closed under $\nabla\Gamma$ have left lifting against $D \to \bullet$, and for each $P \in \mathrm{Proj}(\mathcal{C})$ there is a morphism $P \to D$ which is $\nabla\Gamma$-closed.

Vertical Promotion of Frey's Axioms?

I) A direct approach:

- $\mathcal{C}$ l.c.c. $\mapsto$ locally cartesian closed $(\infty, 1)$-category (slice condition)
- $\mathcal{C}$ exact $\mapsto$ effectivity of relations (groupoid objects)
- ∇ reflective inclusion $\mapsto$ faithful adjoint $(\infty, 1)$-functor
- etc.

However, promoting the behavior of $\mathrm{Proj}(\mathcal{C})$ seemed very difficult (to me).

II) Indirect approach: internalize some model for higher toposes in a well-understood realizability topos. This has been the approach of Frey (2017), who suggests internalizing the cubical set model in $\mathcal{E}ff$.

Partiality in Higher Categories

Def. Higher Restriction

A **restriction ∞-category** $(\mathcal{C}, {}^{-})$ is an $(\infty, 1)-$category $\mathcal{C}$ along with a combinator ${}^{-} : \mathrm{Arr}\mathcal{C} \to \mathrm{Arr}\mathcal{C}$ which assigns to each 1-morphism $f : A \to B$ a 1-morphism $\bar{f} : A \to A$ such that:

i) The composition $f\bar{f}$ exists and there is an 2-morphism $R_1 : f\bar{f} \sim f$

ii) for each 1-morphism g such that the composite $\bar{g}\bar{f}$ exists, then the composite $\bar{f}\bar{g}$ also exists and there is an 2-morphism $R_2 : \bar{g}\bar{f} \sim \bar{f}\bar{g}$

iii) for each 1-morphism g such that the composite $g\bar{f}$ is defined, there is an 2-morphism $R_3 : \overline{g\bar{f}} \sim \bar{g}\bar{f}$

iv) for each 1-morphism g such that gf exists, there is an 2-morphism $R_4 : \bar{g}f \sim f\overline{gf}$

Partiality in Higher Categories

Def. Higher restriction (cont.)

And each of these 2-morphisms is witnessed by homotopy-coherent 2-cells:

$$\begin{tikzcd}
f\bar{f}(x) \arrow[r, dash, "f\bar{f}(p)"] \arrow[d, dash, "R_1(x)"'] \arrow[dr, phantom, "\Downarrow"] & f\bar{f}(y) \arrow[d, dash, "R_1(y)"] \\
f(x) \arrow[r, dash, "\bar{f}(p)"'] & f(y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{g}\bar{f}(x) \arrow[r, dash, "\bar{g}\bar{f}(p)"] \arrow[d, dash, "R_2(x)"'] \arrow[dr, phantom, "\Downarrow"] & \bar{g}\bar{f}(y) \arrow[d, dash, "R_2(y)"] \\
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f\overline{gf}(x) \arrow[r, dash, "f\overline{gf}(p)"'] & f\overline{gf}(y)
\end{tikzcd}$$

where p is now a element derived from higher coherence data.

Remark. If $\mathcal{C}$ were, say, a model category, p would be from a path object for A.

Partiality in Higher Categories

Def. Higher restriction (cont.)

NB - A morphism $f : A \to B$ in a restriction category is called **total** if there is a homotopy $T : \bar{f} \sim id_A$, witnessed by a coherent square:

$$\begin{tikzcd} \bar{f}(x) \arrow[r, dash, "\bar{f}(p)"] \arrow[d, dash, "T(x)"'] \arrow[dr, phantom, "\Downarrow"] & \bar{f}(y) \arrow[d, dash, "T(y)"] \\ x \arrow[r, dash, "p"'] & y \end{tikzcd}$$

- In other words "restriction ∞-categories" are (equivalent to) categories enriched not just over posets, but *directed topological spaces.*
- The exact class of these spaces is not yet clear (d-Spaces? etc.)

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- In other words "restriction ∞-categories" are (equivalent to) categories enriched not just over posets, but *directed topological spaces*.
- The exact class of these spaces is not yet clear (d-Spaces? etc.)
- One thing is extra tantalizing: some form of *concurrency* is naturally appearing as a model of higher partiality.

Higher Turing Categories

Finally, Turing categories will be vertically promoted based on the following lemma:

Lemma.

A cartesian restriction category $\mathcal{C}$ is Turing iff it has an object A with universal application $\bullet : A \times A \to A$, and of which every $X \in \mathcal{C}$ is a retract.

The following is now routine:

Def. Higher Turing category

A *higher Turing category* is a cartesian restriction ∞-category $\mathcal{C}$ with an object A of $\mathcal{C}$ and coherent application $\bullet : A \times A \to A$, such that every object X of $\mathcal{C}$ is a homotopy retract of A.

Example: the nerve of a Turing category.

Univalence vs. the Univalence Axiom

Univalence has two main roles in the use of (book) HoTT: i) as an extensionality principle, ii) as a coherence principle. Its two main issues are well-known:

- It is a statement about computing certain proof witnesses (namely, paths in identity types).

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Much work devoted to formulating univalence as a computational rule within homotopy type theory:

- two-level/two-dimensional type theory (Angiuli, Harper, Favonia, Licata,...)
- cubical type theories (Cohen, Coquand, Huber, Mörtberg,...)

Both De Morgan and Cartesian TT indeed verify univalence!

A Place for Realizability

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- Uemura (2018) provides a counterexample to propositional resizing in a model of cubical assemblies with an impredicative universe satisfying univalence. (This shows the independence of propositional resizing from UA.) But this model is "far from a realizability ∞-topos".

Syntax for Higher Triposes

Let $\mathcal{C}, \mathcal{D}$ be higher Turing categories, and $\mathcal{F} : \mathcal{D} \to \mathcal{C}$ a restriction ∞-functor.

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Suggestion

- Define an **∞-assembly** on $\mathcal{D}$ to be a higher restriction idempotent on $\mathcal{F}(A) \times X \in \mathcal{C}$.
- Morphisms of ∞-assemblies are then **tracked** by $\mathcal{D}$-morphisms $d : A \to B$ s.t.

$$\begin{gathered}
\text{i) } (\mathcal{F}(d) \times f) \circ \alpha \sim \beta \circ (\mathcal{F}(d) \times f) \circ \alpha, \\
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- **Def.** (special case) The functor ∂_∞, an $(\infty, 1)$-Grothendieck fibration, is a **homtopy tripos**.

Higher Partiality Monads?

N. Veltri (2008) defines a certain partial map classifier $D_{\approx}$ on a restriction category, otherwise known to Capretta, Altenkirch...as a *partiality monad*. This construction allows MLTT to deal syntactically with "domains of definition", traditionally not a feature of type theories.

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N. Veltri (2008) defines a certain partial map classifier $D_{\approx}$ on a restriction category, otherwise known to Capretta, Altenkirch...as a *partiality monad*. This construction allows MLTT to deal syntactically with "domains of definition", traditionally not a feature of type theories.

Conjecture

Higher restriction structures as defined here yield a similar construction of a partiality $(\infty, 1)$-monad definable in the syntax of homotopy type theory.

- For **concurrency monads** which generalize Capretta's delay/partiality monad see, e.g. Piróg and Gibbons, *Tracing monadic computations and representing effects*.

This would be a partial map classifier for sections of type families, with domains of definition, over a Martin-Löf universe with identity types.

Comments and Questions are Welcome

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