Toward a Homotopy Tripos for Higher Realizability

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Main Goal

Our simple goal is to discuss the following definition:

Def. Higher Turing category

A higher Turing category is a cartesian restriction ∞ -category \mathcal{C} with an object A of \mathcal{C} and coherent application $\bullet : A \times A \to A$, such that every object X of \mathcal{C} is a homotopy retract of A.

and its relevance to computable interpretations of univalence in HoTT.

Categorifying Recursion Theory

- The notion of *function partiality* is foreign to type theories, both natively and as the internal logics of categories.
- Models of computability as found in recursion theory reflect this fact, e.g. PCAs are models of *untyped* lambda calculi.

Categorical folks interested in computability have therefore introduced many ways to apply intrinsically categorical methods to recursion theory:

- Partial map categories (Longo, Moggi, Robinson, Rosolini...)
- Dominical/recursion categories (Di Paola, Heller, Montagna, Lengyel...)
- The recursive topos (Mulry...)
- Arithmetical universes (Joyal...)
- Realizability toposes (Hyland, Pitts, Johnstone...)
- The effective topos (Hyland...)
- Restriction categories (Cockett, Lack...)

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The minimalism and equationality of restriction categories make them our starting point for "homotopifying" recursion theory with a view towards a realizable interpretation of univalence.

Categorifying Recursion Theory

Def. Restriction category

A restriction category $(\mathcal{C},\bar{})$ is a category \mathcal{C} along with a combinator $\bar{}: \operatorname{Arr}\mathcal{C} \to \operatorname{Arr}\mathcal{C}$ assigning to each arrow $f: A \to B$ in \mathcal{C} an $\bar{f}: A \to A$ such that:

i) $f\bar{f} = f$ ii) for all g : Hom_{\mathcal{C}}(A, C), $\bar{g}\bar{f} = \bar{f}\bar{g}$ iii) for all g : Hom_{\mathcal{C}}(A, C), $\bar{g}\bar{f} = \bar{g}\bar{f}$ iv) for all g : Hom_{\mathcal{C}}(B, C), $\bar{g}f = fg\bar{f}$

NB - A morphism $f : A \to B$ in \mathcal{C} is called **total** if $\overline{f} = id_A$.

- Functors (properly, restriction functors) of restriction categories preserve the partiality structures.
- Objects and total maps of \mathcal{C} form a subcategory $\operatorname{Tot}(C,\bar{}) \hookrightarrow (\mathcal{C},\bar{})$ in this sense.
- Examples: **ParSet**, **Rec**, **ParTop**...

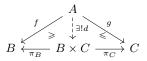
Restriction Categories - Some Properties

Diagrams in a restriction category do not commute equationally, but as inequalities in the poset order induced by restriction. E.g.

■ a **restricted final object 1** for a restriction category (*C*, ⁻) has the universal property:



• and restricted binary products satisfy:



where the projections are total.

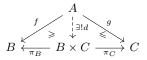
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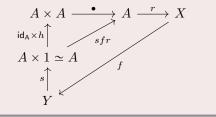
Def.

A restriction category with restricted binary products and a restricted final object is called a **cartesian restriction category**.

Turing (1-)Categories

Def. Turing Category

A **Turing category** is a cartesian restriction category $(\mathcal{C}, \overline{})$ with a fixed object A and morphism $\bullet : A \times A \to A$ having the following universal property: for each \mathcal{C} -morphism $f : X \to Y$ there is a section $s : Y \to A$ and retract $r : A \to X$, along with a total map $h : \mathbf{1} \to A$ satisfying the diagram:



That is, each map in C ia A-computable up to sections and retractions. Note that this commutes equationally; the products shown are restricted products.

Turing (1-)categories

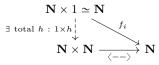
The universal object in \mathcal{C} , the Turing object, should be thought of as Gödel-encoding all maps $f: X \to Y$ in \mathcal{C} via its application $A \times X \to Y$, represented by the global section $h: 1 \to A$.

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Basic examples:

• Rec, of natural numbers $n, m \in \mathbb{N}$ and partial recursive functions $\mathbb{N}^n \to \mathbb{N}^m$, with universal applicative structure:



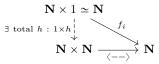
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where the representation is $\langle i, n \rangle = f_i(n)$, the *i*th computable function.

- Conversely, any PCA gives rise to a Turing category, via its computable map category.
- The Karoubi envelope (idempotent splitting) of any Turing category is a Turing category.

Turing Objects as Relative PCAs

Just as the computable map category of a PCA forms a Turing category, a Turing object A of C and its Turing morphism • : $A \times A \rightarrow A$ forms a PCA in C.

Def.

A (relative) PCA A is a combinatory complete partial applicative system in a cartesian restriction category \mathcal{D} .

- Partial applicative system := a morphism : $A \times A \to A$ in \mathcal{D}
- Completeness in \mathcal{D} := finite powers A^n and A-computable morphisms form a well-defined cartesian restriction subcategory of \mathcal{D}

Caveat. Not every PCA in a Turing category C is a Turing object for C.

Note on Formalization

The preceding material (and more), but not the material to follow, has been formalized in Coq by Vinogradova, Felty, and Scott (2018), code available at:

github.com/polinavino/Turing-Category-Formalization

Realizability Toposes from Turing Categories

The Turing object-to-realizability topos construction works much as in the case for classical PCAs, but is again a purely categorical formulation. Let $\mathcal{F} : \mathcal{D} \to \mathcal{C}$ be a restriction functor.

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- Let β be an \mathcal{D} -assembly with object $\mathcal{F}(B) \times Y \in C$. Then a morphism of assemblies $f : \alpha \to \beta$ is a map $f : X \to Y$ in \mathcal{C} s.t. there exists a \mathcal{D} -morphism $d : A \to B$ and:

i)
$$(\mathcal{F}(d) \times f) \circ \alpha = \beta \circ (\mathcal{F}(d) \times f) \circ \alpha$$
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ii)
$$\overline{(\mathsf{id}_{\mathcal{F}(A)} \times f) \circ \alpha} = \overline{(\mathcal{F}(d) \times f) \circ \alpha}.$$

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- These assemblies and their morphisms form a (restriction) category $ASM(\mathcal{F})$
- There is now a forgetful functor $\partial : ASM(\mathcal{F}) \longrightarrow \mathcal{C}$.

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When C and D are Turing categories, ∂ is a fibration, and this fibration is a tripos. In fact, $\partial(ASM(\mathcal{F}))$ is a *realizability tripos* whose internal language defines a realizability topos.

Realizability Toposes: An Extensional Characterization

"Frey's axioms" (2014/2018): a resolution of Johnstone's complaint (c. 2010)?

Theorem (Frey 2014)

A (locally small) category ${\mathcal C}$ is a realizability topos iff:

I) ${\mathcal C}$ is exact and locally cartesian closed,

II) \mathcal{C} has enough projectives and the full subcategory $\operatorname{Proj}(\mathcal{C}) \hookrightarrow \mathcal{C}$ has all finite limits,

III) The global section functor

$$\Gamma := \mathcal{C}(\bullet, -) : \mathcal{C} \longrightarrow \mathbf{Set}$$

has right adjoint $\nabla : \mathbf{Set} \hookrightarrow \mathcal{C}$, a reflective inclusion s.t. $\nabla \Gamma$ preserves finite limits and the idempotent closure operator. The modality ∇ factors through $\operatorname{Proj}(\mathcal{C})$.

IV) Finally, there is a $\nabla\Gamma$ -separated object $D \in \operatorname{Proj}(\mathcal{C})$ s.t. regular epics closed under $\nabla\Gamma$ have left lifting against $D \to \bullet$, and for each $P \in \operatorname{Proj}(\mathcal{C})$ there is a morphism $P \to D$ which is $\nabla\Gamma$ -closed.

Vertical Promotion of Frey's Axioms?

- I) A direct approach:
 - C l.c.c. \mapsto locally cartesian closed (∞ , 1)-category (slice condition)
 - \mathcal{C} exact \mapsto effectivity of relations (groupoid objects)
 - ∇ reflective inclusion \mapsto faithful adjoint $(\infty, 1)$ -functor
 - etc.

However, promoting the behavior of $\operatorname{Proj}(\mathcal{C})$ seemed very difficult (to me).

II) Indirect approach: internalize some model for higher toposes in a well-understood realizability topos. This has been the approach of Frey (2017), who suggests internalizing the cubical set model in $\mathcal{E}ff$.

Def. Higher Restriction

A restriction ∞ -category $(\mathcal{C},\bar{})$ is an $(\infty, 1)$ -category \mathcal{C} along with a combinator $\bar{}: \operatorname{Arr}\mathcal{C} \to \operatorname{Arr}\mathcal{C}$ which assigns to each 1-morphism $f: A \to B$ a 1-morphism $\bar{f}: A \to A$ such that:

i) The composition $f\bar{f}$ exists and there is an 2-morphism $R_1: f\bar{f} \sim f$

ii) for each 1-morphism g such that the composite $\bar{g}\bar{f}$ exists, then the composite $\bar{f}\bar{g}$ also exists and there is an 2-morphism $R_2: \bar{g}\bar{f} \sim \bar{f}\bar{g}$

iii) for each 1-morphism g such that the composite $g\bar{f}$ is defined, there is an 2-morphism $R_3: \overline{g\bar{f}} \sim \bar{g}\bar{f}$

iv) for each 1-morphism g such that gf exists, there is an 2-morphism $R_4:\bar{g}f\sim f\overline{gf}$

Def. Higher restriction (cont.)

And each of these 2-morphisms is witnessed by homotopy-coherent 2-cells:

where p is now a element derived from higher coherence data.

Remark. If \mathcal{C} were, say, a model category, p would be from a path object for A.

Def. Higher restriction (cont.)

NB - A morphism $f : A \to B$ in a restriction category is called **total** if there is a homotopy $T : \overline{f} \sim id_A$, witnessed by a coherent square:

$$\begin{array}{c|c} \bar{f}(x) & \underline{\bar{f}(p)} & \bar{f}(y) \\ \hline f(x) & \downarrow & \downarrow \\ x & \underline{\qquad p \qquad } y \end{array}$$

- In other words "restriction ∞-categories" are (equivalent to) categories enriched not just over posets, but *directed topological spaces*.
- The exact class of these spaces is not yet clear (d-Spaces? etc.)

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- The exact class of these spaces is not yet clear (d-Spaces? etc.)
- One thing is extra tantalizing: some form of *concurrency* is naturally appearing as a model of higher partiality.

Higher Turing Categories

Finally, Turing categories will be vertically promoted based on the following lemma:

Lemma.

A cartesian restriction category C is Turing iff it has an object A with universal application $\bullet : A \times A \to A$, and of which every $X \in C$ is a retract.

The following is now routine:

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A higher Turing category is a cartesian restriction ∞ -category \mathcal{C} with an object A of \mathcal{C} and coherent application $\bullet : A \times A \to A$, such that every object X of \mathcal{C} is a homotopy retract of A.

Example: the nerve of a Turing category.

Univalence vs. the Univalence Axiom

Univalence has two main roles in the use of (book) HoTT: i) as an extensionality principle, ii) as a coherence principle. Its two main issues are well-known:

• It is a statement about computing certain proof witnesses (namely, paths in identity types).

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Much work devoted to formulating univalence as a computational rule within homotopy type theory:

- two-level/two-dimensional type theory (Angiuli, Harper, Favonia, Licata,...)
- cubical type theories (Cohen, Coquand, Huber, Mörtberg,...)

Both De Morgan and Cartesian TT indeed verify univalence!

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In short, in order to have complete computational meaning, we now expect constructions in HoTT, especially paths introduced by univalence, to have an interpretation in an as-yet unknown realizability ∞ -topos. Details are emerging:

■ Uemura (2018) provides a counterexample to propositional resizing in a model of cubical assemblies with an impredicative universe satisfying univalence. (This shows the independence of propositional resizing from UA.) But this model is "far from a realizability ∞-topos".

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Suggestion

- Define an ∞ -assembly on \mathcal{D} to be a higher restriction idempotent on $\mathcal{F}(A) \times X \in \mathcal{C}$.
- Morphisms of ∞ -assemblies are then **tracked** by \mathcal{D} -morphisms $d: A \to B$ s.t.

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■ Def. (special case) The functor ∂_∞, an (∞, 1)-Grothendieck fibration, is a homtopy tripos.

Higher Partiality Monads?

N. Veltri (2008) defines a certain partial map classifier D_{\approx} on a restriction category, otherwise known to Capretta, Altenkirch...as a *partiality monad*. This construction allows MLTT to deal syntactically with "domains of definition", traditionally not a feature of type theories.

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Conjecture

Higher restriction structures as defined here yield a similar construction of a partiality $(\infty, 1)$ -monad definable in the syntax of homotopy type theory.

• For **concurrency monads** which generalize Capretta's delay/partiality monad see, e.g. Piróg and Gibbons, *Tracing monadic computations and representing effects*.

This would be a partial map classifier for sections of type families, with domains of definition, over a Martin-Löf universe with identity types.

Comments and Questions are Welcome

Thank you to Carnegie Mellon University and the HoTT2019 organizing team for making the summer school and conference happen, and the scientific committee for kindly funding members of this conference.