Towards a constructive simplicial model of univalent foundations

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Goal

To define a model of Univalent Foundations that is

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(2) defined in a category homotopically-equivalent to **Top**.

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Univalent Foundations $=$ **ML** $+$ **UA** , where

- **ML** $=$ Martin-Löf type theory with one universe type
- **UA** $=$ Voevodsky's Univalence Axiom

Related work

Cubical approach:

- [BCH], [BCHM], [OP], ... do (1) but not (2).
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Simplicial approach has some advantages:

- more familiar
- uses standard notion of Kan fibration
- straightforward equivalence with **Top**.

Main result

Theorem (Gambino and Henry)**.** Constructively, there exists a comprehension category

$$\begin{array}{ccc} \mathbf{Fib} & \xrightarrow{\quad\chi\quad} & \mathbf{SSet}_{\mathbf{cof}}^{\rightarrow} \\ & \searrow \quad \swarrow_{\mathrm{cod}} & \\ & \mathbf{SSet}_{\mathbf{cof}} & \end{array}$$

with

- all the type constructors of **ML**
- univalence of the universe
- Π-types are weakly stable, other type constructors are pseudo-stable.

$\mathbf{SSet}_{\mathbf{cof}}$ = full subcategory of cofibrant simplicial sets $\subsetneq$ **SSet**

References

[H1] S. Henry
Weak model structures in classical and constructive mathematics
ArXiv, 2018

[H2] S. Henry
A constructive account of the Kan-Quillen model structure and of Kan's Ex^∞ functor
ArXiv, 2019

[GSS] N. Gambino and K. Szumiło and C. Sattler
The constructive Kan-Quillen model structure: two new proofs
ArXiv, 2019

[GH] N. Gambino and S. Henry
Towards a constructive simplicial model of Univalent Foundations
ArXiv, 2019

Outline of the talk

- Review of the classical simplicial model
- Constructive simplicial homotopy theory

Voevodsky's classical simplicial model

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- dependent types = Kan fibrations.

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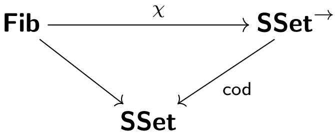

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It gives rise to a strict model via a splitting process.

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(3) There is a Kan fibration $\pi : \tilde{U} \to U$, with U Kan complex, that classifies small Kan fibrations, i.e.

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(4) The Kan fibration $\pi : \tilde{U} \to U$ is univalent.

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- [BCP] shows that (1), (2) require classical logic.
- [GS] fixed (1), (2) by introducing **uniform** Kan fibrations in **SSet**, but this creates problems for (3), (4).

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$$\mathbf{C} = \mathbf{Sat}(I)\,, \qquad \mathbf{W} \cap \mathbf{F} = I^{\pitchfork}$$

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In particular, $\mathbf{F} =$ Kan fibrations. This helps with (3).

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- i is a monomorphism s.t. $\forall n$, $i_n : A_n \to B_n$ is complemented, i.e.

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Note. $\mathbf{C}$ = cofibrations in Reedy wfs generated by the wfs

(Complemented mono, Split epi)

on **Set**.

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Theorem [H2]**.** Constructively, the category **SSet** admits a model structure $(\mathbf{W}, \mathbf{C}, \mathbf{F})$ such that

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- Constructively, not every object is cofibrant: X is cofibrant if and only if degeneracy of simplices in X is decidable.

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Note

- Constructively, not every object is cofibrant: X is cofibrant if and only if degeneracy of simplices in X is decidable.
- Every object X has a cofibrant replacement, given by $\mathbb{L}(X)$ cofibrant and $t : \mathbb{L}(X) \to X$ in $\mathbf{W} \cap \mathbf{C}$.

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Challenge

- stay within the cofibrant fragment.

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is a bijection. Its inverse is written

$$\frac{X \times A \xrightarrow{\quad f \quad} B}{X \xrightarrow{\quad \lambda(f) \quad} B^A}$$

In general, B^A is not cofibrant.

Step 2. Let $\mathbb{L}(B^A)$ be a cofibrant replacement of B^A, with

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Note

- β-rule holds judgementally, η-rule holds propositionally.
- This extends to Π-types.

The universe (I)

Step 1. Construct a Kan fibration $\pi : \tilde{U} \to U$ which classifies small Kan fibrations with cofibrant fibers.

$$U_n = \{ p : A \to \Delta[n] \mid p \text{ small fibration} , A \text{ cofibrant} \}$$

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Step 2.

- Let U_c be a cofibrant replacement of U, with $t : U_c \to U$ in $\mathbf{W} \cap \mathbf{F}$
- Pullback

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□

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- Equivalence extension property
- Diagram-chasing, using 3-for-2 for $\mathbf{W}$.

Coherence issues

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None of the known strictification methods seems to apply constructively.

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- A simplicial type theory extracted from the comprehension category, in which univalence axiom is provable.

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  - Model structure on simplicial sheaves $[\Delta^{\mathrm{op}}, \mathcal{E}]$
  - Connections to higher topos theory
- A simplicial type theory extracted from the comprehension category, in which univalence axiom is provable.

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