First-Order Homotopical Logic

Joseph Helfer, Stanford Univeristy

HoTT 2019
August 12, 2019

Outline

Outline

Consider the diagram...

$$
\begin{tikzcd}
\text{MLTT} \arrow[rr, "\text{Voevodsky-Awodey-Warren-Kapulkin-Lumsdaine-}\cdots"] & & \text{Simplicial Sets} \\
\\
\text{IFOL} \arrow[uu, "\text{Martin-Löf 1972}"]
\end{tikzcd}
$$

Consider the diagram...

$$\begin{tikzcd}
\text{MLTT} \arrow[rr, "\text{Voevodsky-Awodey-Warren-Kapulkin-Lumsdaine-}\cdots"] & & \text{Simplicial Sets} \\
\text{IFOL} \arrow[u, "\text{Martin-Löf 1972}"] \arrow[urr, "\text{First-order homotopical logic}"] & &
\end{tikzcd}$$

Outline

Propositions as objects of **C**

Propositions as objects of **C**

Fix a first-order language $\mathcal{L}$

Propositions as objects of $\mathbf{C}$

Fix a first-order language $\mathcal{L}$ (many-sorted, no relation symbols).

Propositions as objects of $\mathbf{C}$

Fix a first-order language $\mathcal{L}$ (many-sorted, no relation symbols).
Given a category $\mathbf{C}$ with finite products, we can

Propositions as objects of $\mathbf{C}$

Fix a first-order language $\mathcal{L}$ (many-sorted, no relation symbols).
Given a category $\mathbf{C}$ with finite products, we can

- Define the notion of a $\mathbf{C}$-structure M for $\mathcal{L}$

Propositions as objects of $\mathbf{C}$

Fix a first-order language $\mathcal{L}$ (many-sorted, no relation symbols).
Given a category $\mathbf{C}$ with finite products, we can

- Define the notion of a $\mathbf{C}$-structure M for $\mathcal{L}$
- Define $M(\Gamma) \in \mathrm{Ob}\,\mathbf{C}$ for each *context* (=seq. of variables) Γ

Propositions as objects of $\mathbf{C}$

Fix a first-order language $\mathcal{L}$ (many-sorted, no relation symbols). Given a category $\mathbf{C}$ with finite products, we can

- Define the notion of a $\mathbf{C}$-structure M for $\mathcal{L}$
- Define $M(\Gamma) \in \mathrm{Ob}\,\mathbf{C}$ for each *context* (=seq. of variables) Γ
- Define $M_\Gamma(t) : M(\Gamma) \to M(B)$ for each term t of $\mathcal{L}$ of sort B with free variables in Γ

Propositions as objects of **C**

Fix a first-order language $\mathcal{L}$ (many-sorted, no relation symbols).
Given a category **C** with finite products, we can

- Define the notion of a **C**-structure M for $\mathcal{L}$
- Define $M(\Gamma) \in \mathrm{Ob}\,\mathbf{C}$ for each *context* (=seq. of variables) Γ
- Define $M_\Gamma(t) : M(\Gamma) \to M(B)$ for each term t of $\mathcal{L}$ of sort B with free variables in Γ

The "Propositions-as-Objects-of-**C**" semantics

Propositions as objects of $\mathbf{C}$

Fix a first-order language $\mathcal{L}$ (many-sorted, no relation symbols). Given a category $\mathbf{C}$ with finite products, we can

- Define the notion of a $\mathbf{C}$-structure M for $\mathcal{L}$
- Define $M(\Gamma) \in \mathrm{Ob}\,\mathbf{C}$ for each *context* (=seq. of variables) Γ
- Define $M_\Gamma(t) : M(\Gamma) \to M(B)$ for each term t of $\mathcal{L}$ of sort B with free variables in Γ

The "Propositions-as-Objects-of-$\mathbf{C}$" semantics should assign to each formula ϕ with free variables in Γ

Propositions as objects of $\mathbf{C}$

Fix a first-order language $\mathcal{L}$ (many-sorted, no relation symbols). Given a category $\mathbf{C}$ with finite products, we can

- Define the notion of a $\mathbf{C}$-structure M for $\mathcal{L}$
- Define $M(\Gamma) \in \mathrm{Ob}\,\mathbf{C}$ for each *context* (=seq. of variables) Γ
- Define $M_\Gamma(t) : M(\Gamma) \to M(B)$ for each term t of $\mathcal{L}$ of sort B with free variables in Γ

The "Propositions-as-Objects-of-$\mathbf{C}$" semantics should assign to each formula ϕ with free variables in Γ an object $M_\Gamma(\phi) \in \mathrm{Ob}(\mathbf{C}/M(\Gamma))$

Propositions as objects of $\mathbf{C}$

Fix a first-order language $\mathcal{L}$ (many-sorted, no relation symbols). Given a category $\mathbf{C}$ with finite products, we can

- Define the notion of a $\mathbf{C}$-structure M for $\mathcal{L}$
- Define $M(\Gamma) \in \mathrm{Ob}\,\mathbf{C}$ for each *context* (=seq. of variables) Γ
- Define $M_\Gamma(t) : M(\Gamma) \to M(B)$ for each term t of $\mathcal{L}$ of sort B with free variables in Γ

The "Propositions-as-Objects-of-$\mathbf{C}$" semantics should assign to each formula ϕ with free variables in Γ an object $M_\Gamma(\phi) \in \mathrm{Ob}(\mathbf{C}/M(\Gamma))$ (a "family of objects" over $M(\Gamma)$).

Propositions as objects of $\mathbf{C}$

Possible if $\mathbf{C}$ is locally cartesian closed and has finite coproducts.

Propositions as objects of **C**

Possible if **C** is locally cartesian closed and has finite coproducts.

ϕ	$M_\Gamma(\phi)$
$P \wedge Q$	$M_\Gamma(P) \times M_\Gamma(Q)$
$P \vee Q$	$M_\Gamma(P) + M_\Gamma(Q)$
$P \Rightarrow Q$	$M_\Gamma(Q)^{M_\Gamma(P)}$
$(\forall x \in A)P$	$\prod_{\pi : M(\Gamma \cup \{x\}) \to M(\Gamma)} M_\Gamma(P)$
$(\exists x \in A)P$	$\sum_{\pi : M(\Gamma \cup \{x\}) \to M(\Gamma)} M_\Gamma(P)$
$\top$	$\mathbf{1}_{\mathbf{C}/M(\Gamma)}$
$\bot$	$\mathbf{0}_{\mathbf{C}/M(\Gamma)}$

Equality

How do we interpret equality?

Equality

How do we interpret equality?

$$\begin{tikzcd} M(B) \arrow[d, "\Delta_{M(B)}"] \\ M(B) \times M(B) \end{tikzcd}$$

Equality

How do we interpret equality?

$M_{(x,y)}(x = y)$

$$M(B) \xrightarrow{\Delta_{M(B)}} M(B) \times M(B)$$

Equality

How do we interpret equality?

$$\begin{array}{c} M(B) \\ \big\downarrow \Delta_{M(B)} \\ M(B) \times M(B) \end{array}$$

Equality

How do we interpret equality?

$$
\begin{tikzcd}
 & M(B) \arrow[d, "\Delta_{M(B)}"] \\
M(\Gamma) \arrow[r, "\langle M(s), M(t) \rangle"] & M(B) \times M(B)
\end{tikzcd}
$$

Equality

How do we interpret equality?

$$\begin{tikzcd}
\cdot \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & M(B) \arrow[d, "\Delta_{M(B)}"] \\
M(\Gamma) \arrow[rr, "{\langle M(s), M(t) \rangle}"] & & M(B) \times M(B)
\end{tikzcd}$$

Equality

How do we interpret equality?

$M_\Gamma(s = t)$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M(B) \arrow[d, "\Delta_{M(B)}"] \\
M(\Gamma) \arrow[r, "\langle M(s), M(t) \rangle"] & M(B) \times M(B)
\end{tikzcd}$$

Propositions as spaces

Now take $\mathbf{C} =$ "spaces" (e.g., simplicial sets, which is LCCC).

Propositions as spaces

Now take $\mathbf{C} =$ "spaces" (e.g., simplicial sets, which is LCCC).
But

Propositions as spaces

Now take $\mathbf{C} =$ "spaces" (e.g., simplicial sets, which is LCCC).
But, following Voevodsky-Awodey-Warren-Kapulkin-Lumsdaine-$\cdots$,

Propositions as spaces

Now take $\mathbf{C} =$ "spaces" (e.g., simplicial sets, which is LCCC).
But, following Voevodsky-Awodey-Warren-Kapulkin-Lumsdaine-$\cdots$, change the interpretation of *equality*!

Propositions as spaces

Now take $\mathbf{C} =$ "spaces" (e.g., simplicial sets, which is LCCC).
But, following Voevodsky-Awodey-Warren-Kapulkin-Lumsdaine-$\cdots$, change the interpretation of *equality*!

- Equality should be "paths", not "identity"

Propositions as spaces

Now take $\mathbf{C} =$ "spaces" (e.g., simplicial sets, which is LCCC).
But, following Voevodsky-Awodey-Warren-Kapulkin-Lumsdaine-$\cdots$, change the interpretation of *equality*!

- Equality should be "paths", not "identity"
- Instead of $\Delta : X \to X \times X$, use the "path-space fibration" $X^I \to X \times X$.

Propositions as spaces

Now take $\mathbf{C} =$ "spaces" (e.g., simplicial sets, which is LCCC).
But, following Voevodsky-Awodey-Warren-Kapulkin-Lumsdaine-$\cdots$, change the interpretation of *equality*!

- Equality should be "paths", not "identity"
- Instead of $\Delta : X \to X \times X$, use the "path-space fibration" $X^I \to X \times X$.

End of definition of the *homotopical semantics for first-order logic*.

Propositions as spaces

Now take $\mathbf{C} =$ "spaces" (e.g., simplicial sets, which is LCCC).
But, following Voevodsky-Awodey-Warren-Kapulkin-Lumsdaine-$\cdots$, change the interpretation of *equality*!

- Equality should be "paths", not "identity"
- Instead of $\Delta : X \to X \times X$, use the "path-space fibration" $X^I \to X \times X$.

End of definition of the *homotopical semantics for first-order logic*.

For homotopical semantics in *topological spaces*, start with an $\mathcal{L}$-structure in **Top** and apply $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$.

Propositions as spaces

Now take $\mathbf{C}$ = "spaces" (e.g., simplicial sets, which is LCCC).
But, following Voevodsky-Awodey-Warren-Kapulkin-Lumsdaine-$\cdots$, change the interpretation of *equality*!

- Equality should be "paths", not "identity"
- Instead of $\Delta : X \to X \times X$, use the "path-space fibration" $X^I \to X \times X$.

End of definition of the *homotopical semantics for first-order logic*.

For homotopical semantics in *topological spaces*, start with an $\mathcal{L}$-structure in **Top** and apply Sing : **Top** $\to$ **sSet**.

Let us say $M \vDash \phi$ if $M(\phi) \neq \emptyset$.

Examples

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$.

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if f and g are homotopic.

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if f and g are homotopic.

Let $\mathcal{L} = (A, \circ : A \times A \to A)$.

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if f and g are homotopic.

Let $\mathcal{L} = (A, \circ : A \times A \to A)$. Then $M = (M, \circ)$ satisfies

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if f and g are homotopic.

Let $\mathcal{L} = (A, \circ : A \times A \to A)$. Then $M = (M, \circ)$ satisfies $\forall x, y, z[x \circ (y \circ z) = (x \circ y) \circ z]$ if and only if

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if f and g are homotopic.

Let $\mathcal{L} = (A, \circ : A \times A \to A)$. Then $M = (M, \circ)$ satisfies $\forall x, y, z[x \circ (y \circ z) = (x \circ y) \circ z]$ if and only if $\circ$ is *homotopy-associative*.

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if f and g are homotopic.

Let $\mathcal{L} = (A, \circ : A \times A \to A)$. Then $M = (M, \circ)$ satisfies $\forall x, y, z[x \circ (y \circ z) = (x \circ y) \circ z]$ if and only if $\circ$ is *homotopy-associative*.

Let $\mathcal{L} = (A)$.

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if f and g are homotopic.

Let $\mathcal{L} = (A, \circ : A \times A \to A)$. Then $M = (M, \circ)$ satisfies $\forall x, y, z[x \circ (y \circ z) = (x \circ y) \circ z]$ if and only if $\circ$ is *homotopy-associative*.

Let $\mathcal{L} = (A)$. Then $M = (X)$ satisfies

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if f and g are homotopic.

Let $\mathcal{L} = (A, \circ : A \times A \to A)$. Then $M = (M, \circ)$ satisfies $\forall x, y, z[x \circ (y \circ z) = (x \circ y) \circ z]$ if and only if $\circ$ is *homotopy-associative*.

Let $\mathcal{L} = (A)$. Then $M = (X)$ satisfies $\exists x \forall y(x = y)$ if and only if

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if f and g are homotopic.

Let $\mathcal{L} = (A, \circ : A \times A \to A)$. Then $M = (M, \circ)$ satisfies $\forall x, y, z[x \circ (y \circ z) = (x \circ y) \circ z]$ if and only if $\circ$ is *homotopy-associative*.

Let $\mathcal{L} = (A)$. Then $M = (X)$ satisfies $\exists x \forall y(x = y)$ if and only if X is *contractible*.

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if f and g are homotopic.

Let $\mathcal{L} = (A, \circ : A \times A \to A)$. Then $M = (M, \circ)$ satisfies $\forall x, y, z[x \circ (y \circ z) = (x \circ y) \circ z]$ if and only if $\circ$ is *homotopy-associative*.

Let $\mathcal{L} = (A)$. Then $M = (X)$ satisfies $\exists x \forall y(x = y)$ if and only if X is *contractible*.

Let $\mathcal{L} = (A, B, f : A \to B)$.

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if f and g are homotopic.

Let $\mathcal{L} = (A, \circ : A \times A \to A)$. Then $M = (M, \circ)$ satisfies $\forall x, y, z[x \circ (y \circ z) = (x \circ y) \circ z]$ if and only if $\circ$ is *homotopy-associative*.

Let $\mathcal{L} = (A)$. Then $M = (X)$ satisfies $\exists x \forall y(x = y)$ if and only if X is *contractible*.

Let $\mathcal{L} = (A, B, f : A \to B)$. Then $M = (X, Y, f)$ satisfies

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if f and g are homotopic.

Let $\mathcal{L} = (A, \circ : A \times A \to A)$. Then $M = (M, \circ)$ satisfies $\forall x, y, z[x \circ (y \circ z) = (x \circ y) \circ z]$ if and only if $\circ$ is *homotopy-associative*.

Let $\mathcal{L} = (A)$. Then $M = (X)$ satisfies $\exists x \forall y(x = y)$ if and only if X is *contractible*.

Let $\mathcal{L} = (A, B, f : A \to B)$. Then $M = (X, Y, f)$ satisfies $\forall y \in B\ \exists! x \in A\ [f(x) = y]$ if and only if

Examples

Let $\mathcal{L} = (A, B, f : A \to B, g : A \to B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if f and g are homotopic.

Let $\mathcal{L} = (A, \circ : A \times A \to A)$. Then $M = (M, \circ)$ satisfies $\forall x, y, z[x \circ (y \circ z) = (x \circ y) \circ z]$ if and only if $\circ$ is *homotopy-associative*.

Let $\mathcal{L} = (A)$. Then $M = (X)$ satisfies $\exists x \forall y(x = y)$ if and only if X is *contractible*.

Let $\mathcal{L} = (A, B, f : A \to B)$. Then $M = (X, Y, f)$ satisfies $\forall y \in B\ \exists! x \in A\ [f(x) = y]$ if and only if f is a *homotopy-equivalence*.

Outline

Properties

Some properties one might expect/hope for:

Properties

Some properties one might expect/hope for:

- Soundness

Properties

Some properties one might expect/hope for:

- Soundness (i.e., $\vdash \phi$ implies $M \vDash \phi$)

Properties

Some properties one might expect/hope for:

- Soundness (i.e., $\vdash \phi$ implies $M \vDash \phi$)
  - For intuitionistic propositional logic

Properties

Some properties one might expect/hope for:

- Soundness (i.e., $\vdash \phi$ implies $M \vDash \phi$)
  - For intuitionistic propositional logic
  - For equality

Properties

Some properties one might expect/hope for:

- Soundness (i.e., $\vdash \phi$ implies $M \vDash \phi$)
    - For intuitionistic propositional logic
    - For equality
    - For LEM?

Properties

Some properties one might expect/hope for:

- Soundness (i.e., $\vdash \phi$ implies $M \vDash \phi$)
  - For intuitionistic propositional logic
  - For equality
  - For LEM? **Counterexample:**

Properties

Some properties one might expect/hope for:

- Soundness (i.e., $\vdash \phi$ implies $M \vDash \phi$)
  - For intuitionistic propositional logic
  - For equality
  - For LEM? **Counterexample:**
    Consider $\exists x \forall y (\neg\neg x = y) \Rightarrow \exists x \forall y (x = y)$.

Properties

Some properties one might expect/hope for:

- Soundness (i.e., $\vdash \phi$ implies $M \vDash \phi$)
  - For intuitionistic propositional logic
  - For equality
  - For LEM? **Counterexample:**
    Consider $\exists x \forall y (\neg\neg x = y) \Rightarrow \exists x \forall y (x = y)$.
    A space X satisfies

Properties

Some properties one might expect/hope for:

- Soundness (i.e., $\vdash \phi$ implies $M \vDash \phi$)
  - For intuitionistic propositional logic
  - For equality
  - For LEM? **Counterexample:**
    Consider $\exists x \forall y (\neg\neg x = y) \Rightarrow \exists x \forall y (x = y)$.
    A space X satisfies
    - the antecedent if and only if X is path-connected.

Properties

Some properties one might expect/hope for:

- Soundness (i.e., $\vdash \phi$ implies $M \models \phi$)
  - For intuitionistic propositional logic
  - For equality
  - For LEM? **Counterexample:**
    Consider $\exists x \forall y (\neg\neg x = y) \Rightarrow \exists x \forall y (x = y)$.
    A space X satisfies
    - the antecedent if and only if X is path-connected.
    - the consequent if and only if X is contractible.

Properties

Some properties one might expect/hope for:

- Soundness (i.e., $\vdash \phi$ implies $M \vDash \phi$)
  - For intuitionistic propositional logic
  - For equality
  - For LEM? **Counterexample:**
    Consider $\exists x \forall y (\neg\neg x = y) \Rightarrow \exists x \forall y (x = y)$.
    A space X satisfies
    - the antecedent if and only if X is path-connected.
    - the consequent if and only if X is contractible.
- Completeness

Properties

Some properties one might expect/hope for:

- Soundness (i.e., $\vdash \phi$ implies $M \vDash \phi$)
  - For intuitionistic propositional logic
  - For equality
  - For LEM? **Counterexample:**
    Consider $\exists x \forall y(\neg\neg x = y) \Rightarrow \exists x \forall y(x = y)$.
    A space X satisfies
    - the antecedent if and only if X is path-connected.
    - the consequent if and only if X is contractible.
- Completeness
  - ???

Properties

Some properties one might expect/hope for:

- Soundness (i.e., $\vdash \phi$ implies $M \vDash \phi$)
  - For intuitionistic propositional logic
  - For equality
  - For LEM? **Counterexample:**
    Consider $\exists x \forall y (\neg\neg x = y) \Rightarrow \exists x \forall y (x = y)$.
    A space X satisfies
    - the antecedent if and only if X is path-connected.
    - the consequent if and only if X is contractible.
- Completeness
  - ???
- Homotopy-invariance

Invariance

Invariance

The **classical (Tarskian) semantics** is *isomorphism-invariant*.

Invariance

The **classical (Tarskian) semantics** is *isomorphism-invariant*.

- I.e., for *isomorphic* $\mathcal{L}$-structures, M and N and a sentence ϕ:

Invariance

The **classical (Tarskian) semantics** is *isomorphism-invariant*.

- I.e., for *isomorphic* $\mathcal{L}$-structures, M and N and a sentence ϕ:

$$M \vDash \phi \quad \text{if and only if} \quad N \vDash \phi$$

Invariance

The **classical (Tarskian) semantics** is *isomorphism-invariant*.

- I.e., for *isomorphic* $\mathcal{L}$-structures, M and N and a sentence ϕ:

$$M \vDash \phi \quad \text{if and only if} \quad N \vDash \phi$$

(and an analogous property for non-closed formulas).

Invariance

The **classical (Tarskian) semantics** is *isomorphism-invariant*.

- I.e., for *isomorphic* $\mathcal{L}$-structures, M and N and a sentence ϕ:

$$M \vDash \phi \quad \text{if and only if} \quad N \vDash \phi$$

  (and an analogous property for non-closed formulas).
- Easy proof by induction on ϕ

Invariance

The **classical (Tarskian) semantics** is *isomorphism-invariant*.

- I.e., for *isomorphic* $\mathcal{L}$-structures, M and N and a sentence ϕ:

$$M \vDash \phi \quad \text{if and only if} \quad N \vDash \phi$$

  (and an analogous property for non-closed formulas).
- Easy proof by induction on ϕ

For **Propositions-as-objects-of-C**, this can be strengthened:

Invariance

The **classical (Tarskian) semantics** is *isomorphism-invariant*.

- I.e., for *isomorphic* $\mathcal{L}$-structures, M and N and a sentence ϕ:

$$M \vDash \phi \quad \text{if and only if} \quad N \vDash \phi$$

  (and an analogous property for non-closed formulas).
- Easy proof by induction on ϕ

For **Propositions-as-objects-of-C**, this can be strengthened:

- For M, N, and ϕ as above

Invariance

The **classical (Tarskian) semantics** is *isomorphism-invariant*.

- I.e., for *isomorphic* $\mathcal{L}$-structures, M and N and a sentence ϕ:

$$M \models \phi \quad \text{if and only if} \quad N \models \phi$$

  (and an analogous property for non-closed formulas).

- Easy proof by induction on ϕ

For **Propositions-as-objects-of-C**, this can be strengthened:

- For M, N, and ϕ as above

$$M(\phi) \text{ and } N(\phi) \text{ are ("canonically") isomorphic}$$

Invariance

The **classical (Tarskian) semantics** is *isomorphism-invariant*.

- I.e., for *isomorphic* $\mathcal{L}$-structures, M and N and a sentence ϕ:

$$M \vDash \phi \quad \text{if and only if} \quad N \vDash \phi$$

  (and an analogous property for non-closed formulas).
- Easy proof by induction on ϕ

For **Propositions-as-objects-of-C**, this can be strengthened:

- For M, N, and ϕ as above

$$M(\phi) \text{ and } N(\phi) \text{ are (“canonically”) isomorphic}$$

  (and again something for non-closed formulas).

Invariance

The **classical (Tarskian) semantics** is *isomorphism-invariant*.

- I.e., for *isomorphic* $\mathcal{L}$-structures, M and N and a sentence ϕ:

$$M \models \phi \quad \text{if and only if} \quad N \models \phi$$

  (and an analogous property for non-closed formulas).
- Easy proof by induction on ϕ

For **Propositions-as-objects-of-C**, this can be strengthened:

- For M, N, and ϕ as above

$$M(\phi) \text{ and } N(\phi) \text{ are ("canonically") isomorphic}$$

  (and again something for non-closed formulas).
- Again, easy inductive proof

Homotopy-invariance

The homotopical semantics satisfy an even stronger(*) property:

Homotopy-invariance

The homotopical semantics satisfy an even stronger(*) property:

- Given *homotopy-equivalent* structures M and N

Homotopy-invariance

The homotopical semantics satisfy an even stronger(*) property:

- Given *homotopy-equivalent* structures M and N

  $M(\phi)$ and $N(\phi)$ are *homotopy-equivalent*

Homotopy-invariance

The homotopical semantics satisfy an even stronger(*) property:

- Given *homotopy-equivalent* structures M and N

  $M(\phi)$ and $N(\phi)$ are *homotopy-equivalent*

- Here, M and N are *homotopy equivalent*

Homotopy-invariance

The homotopical semantics satisfy an even stronger(*) property:

- Given *homotopy-equivalent* structures M and N

  $$M(\phi) \text{ and } N(\phi) \text{ are } \textit{homotopy-equivalent}$$

- Here, M and N are *homotopy equivalent* if there are homotopy-equivalences $h_A : M(A) \simeq N(A)$

Homotopy-invariance

The homotopical semantics satisfy an even stronger(*) property:

- Given *homotopy-equivalent* structures M and N

  $$M(\phi) \text{ and } N(\phi) \text{ are } \textit{homotopy-equivalent}$$

- Here, M and N are *homotopy equivalent* if there are homotopy-equivalences $h_A : M(A) \simeq N(A)$ s.t.

$$\begin{tikzcd}
M(A) \times M(B) \arrow[r, "h_A \times h_B"] \arrow[d, "M(f)"'] & N(A) \times N(B) \arrow[d, "N(f)"] \\
M(C) \arrow[r, "h_C"] & N(C)
\end{tikzcd}$$

etc. commute *up to homotopy*.

Homotopy-invariance

The homotopical semantics satisfy an even stronger(*) property:

- Given *homotopy-equivalent* structures M and N

  $$M(\phi) \text{ and } N(\phi) \text{ are } \textit{homotopy-equivalent}$$

- Here, M and N are *homotopy equivalent* if there are homotopy-equivalences $h_A : M(A) \simeq N(A)$ s.t.

  $$\begin{tikzcd} M(A) \times M(B) \arrow[r, "h_A \times h_B"] \arrow[d, "M(f)"'] & N(A) \times N(B) \arrow[d, "N(f)"] \\ M(C) \arrow[r, "h_C"] & N(C) \end{tikzcd}$$

  etc. commute *up to homotopy*.

- Can be proven by induction, but not so easily.

Homotopy-invariance

The homotopical semantics satisfy an even stronger(*) property:

- Given *homotopy-equivalent* structures M and N

  $M(\phi)$ and $N(\phi)$ are *homotopy-equivalent*

- Here, M and N are *homotopy equivalent* if there are homotopy-equivalences $h_A : M(A) \simeq N(A)$ s.t.

$$\begin{tikzcd} M(A) \times M(B) \arrow[r, "h_A \times h_B"] \arrow[d, "M(f)"'] & N(A) \times N(B) \arrow[d, "N(f)"] \\ M(C) \arrow[r, "h_C"] & N(C) \end{tikzcd}$$

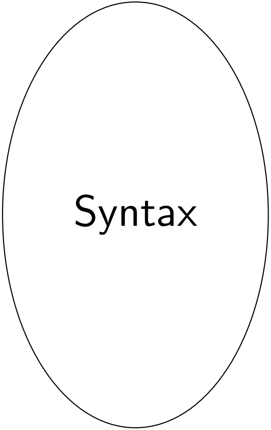
  etc. commute *up to homotopy*.

- Can be proven by induction, but not so easily.
- There is a more conceptual (and general) proof using "fibrational" semantics

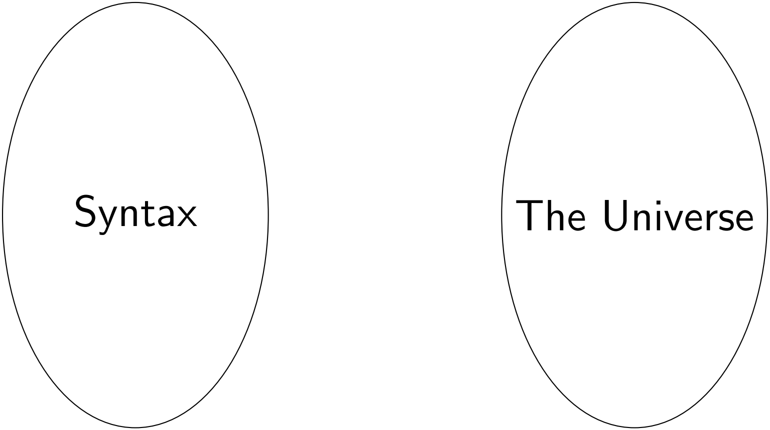
Outline

Functorial semantics (Lawvere)

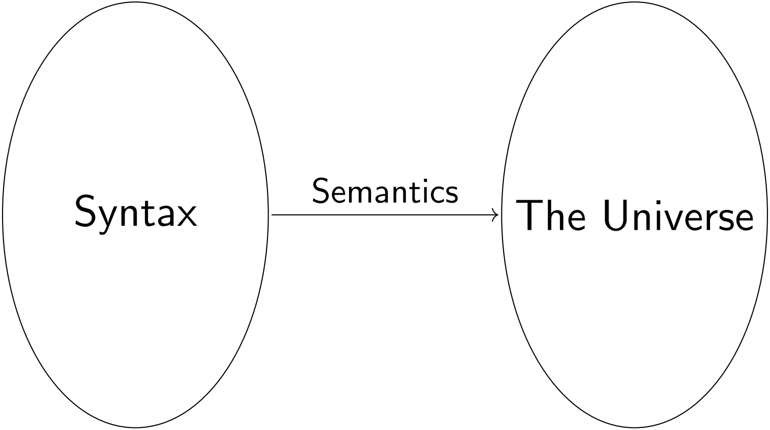
Functorial semantics (Lawvere)



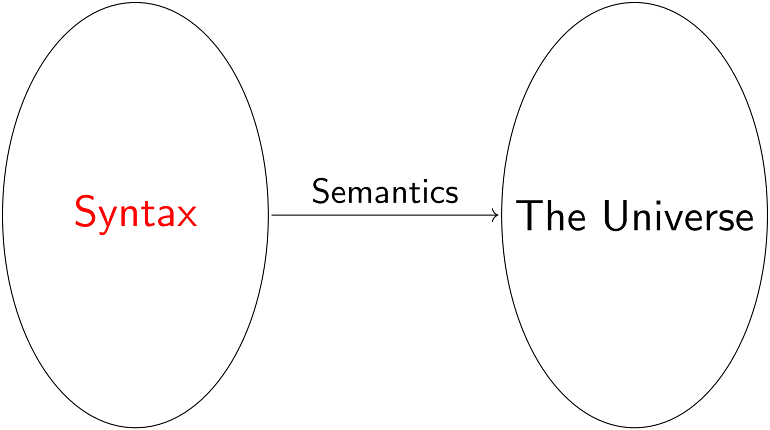
Functorial semantics (Lawvere)



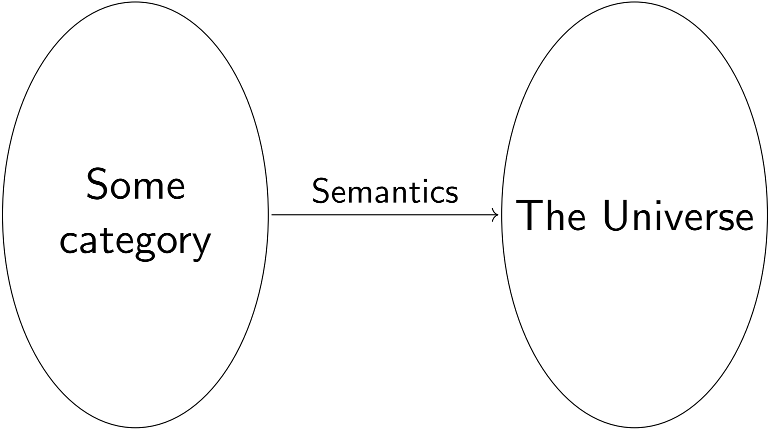
Functorial semantics (Lawvere)



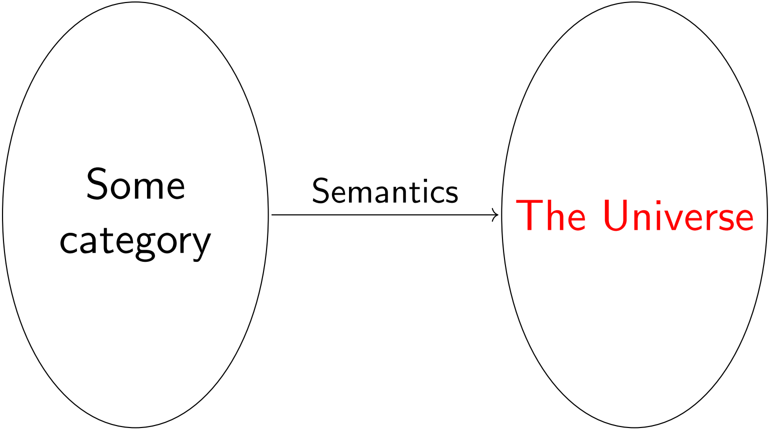
Functorial semantics (Lawvere)



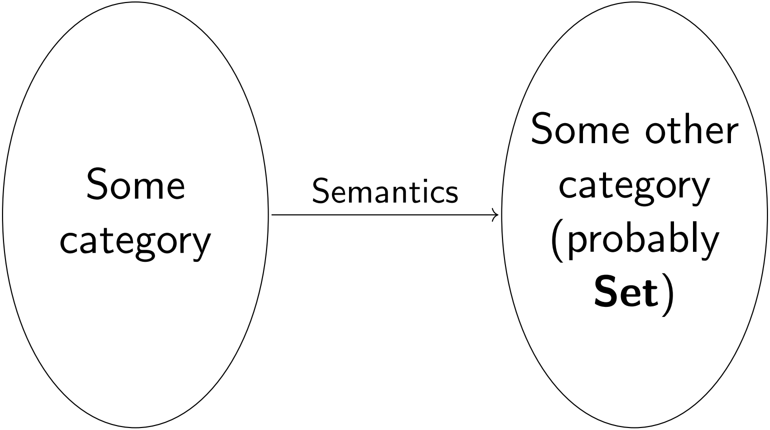
Functorial semantics (Lawvere)



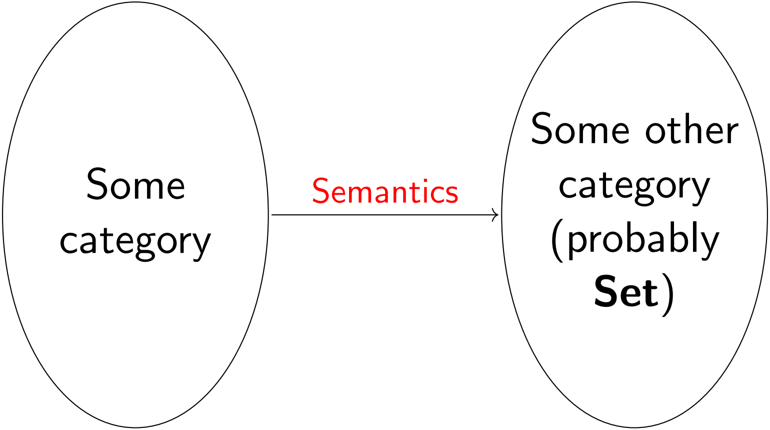
Functorial semantics (Lawvere)



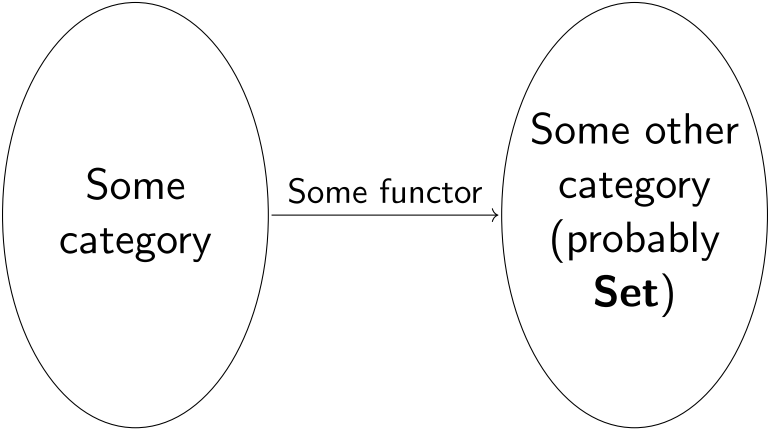
Functorial semantics (Lawvere)



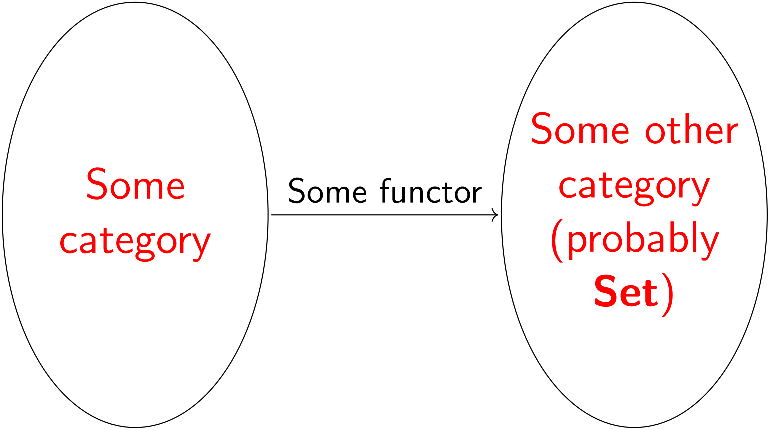
Functorial semantics (Lawvere)



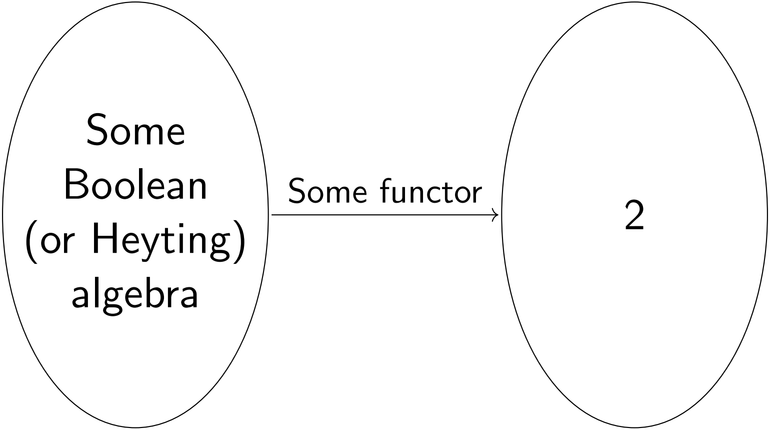
Functorial semantics (Lawvere)



Functorial semantics (Lawvere)



Functorial semantics (Lawvere)



Freeness

Freeness

The Boolean/Heyting algebra B_Σ of propositions over a set Σ of atoms is *free*.

Freeness

The Boolean/Heyting algebra B_Σ of propositions over a set Σ of atoms is *free*.

$$\begin{tikzcd} B_\Sigma & 2 \\ \Sigma \arrow[u, hook] \end{tikzcd}$$

Freeness

The Boolean/Heyting algebra B_Σ of propositions over a set Σ of atoms is *free*.

$$
\begin{tikzcd}
B_\Sigma & 2 \\
\Sigma \arrow[u, hook] \arrow[ur]
\end{tikzcd}
$$

Freeness

The Boolean/Heyting algebra B_Σ of propositions over a set Σ of atoms is *free*.

$$\begin{tikzcd} B_\Sigma \arrow[r, dashed] & 2 \\ \Sigma \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

Freeness

The Boolean/Heyting algebra B_Σ of propositions over a set Σ of atoms is *free*.

$$\begin{tikzcd} B_\Sigma \arrow[r, dashed] & 2 \\ \Sigma \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

Instead, can take the free "non-posetal Heyting algebra" (CCC w/ finite coproducts) $\mathbf{C}_\Sigma$.

Freeness

The Boolean/Heyting algebra B_Σ of propositions over a set Σ of atoms is *free*.

$$\begin{tikzcd} B_\Sigma \arrow[r, dashed] & 2 \\ \Sigma \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

Instead, can take the free "non-posetal Heyting algebra" (CCC w/ finite coproducts) $\mathbf{C}_\Sigma$. This is Lambek's "category of proofs".

Freeness

The Boolean/Heyting algebra B_Σ of propositions over a set Σ of atoms is *free*.

$$\begin{tikzcd} B_\Sigma \arrow[r, dashed] & 2 \\ \Sigma \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

Instead, can take the free "non-posetal Heyting algebra" (CCC w/ finite coproducts) $\mathbf{C}_\Sigma$. This is Lambek's "category of proofs".
This gives the "Propositions-as-objects-of $\mathbf{C}$" semantics:

The Boolean/Heyting algebra B_Σ of propositions over a set Σ of atoms is *free*.

$$\begin{tikzcd} B_\Sigma \arrow[r, dashed] & 2 \\ \Sigma \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

Instead, can take the free "non-posetal Heyting algebra" (CCC w/ finite coproducts) $\mathbf{C}_\Sigma$. This is Lambek's "category of proofs". This gives the "Propositions-as-objects-of $\mathbf{C}$" semantics:

$$\begin{tikzcd} \mathbf{C}_\Sigma \arrow[r, dashed] & \mathbf{C} \\ \Sigma \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

Invariance

Invariance

Using the "categorical" universal property of $\mathbf{C}_\Sigma$

Invariance

Using the "categorical" universal property of $\mathbf{C}_\Sigma$, we obtain an "isomorphism invariance" property:

Using the "categorical" universal property of $\mathbf{C}_\Sigma$, we obtain an "isomorphism invariance" property:

$$\begin{tikzcd}
\mathbf{C}_\sigma \arrow[r, bend left] \arrow[r] & \mathbf{Set} \\
\Sigma \arrow[u, hook] \arrow[ur] \arrow[ur, bend right] \arrow[ur, phantom, shift right=3ex, "\cong"] &
\end{tikzcd}$$

Invariance

Using the "categorical" universal property of $\mathbf{C}_\Sigma$, we obtain an "isomorphism invariance" property:

$$
\begin{tikzcd}
\mathbf{C}_\sigma \arrow[r, bend left] \arrow[r] \arrow[r, phantom, shift left=3, "\cong"] & \mathbf{Set} \\
\Sigma \arrow[u, hook] \arrow[ur] \arrow[ur, bend right] \arrow[ur, phantom, shift right=2, "\cong"] &
\end{tikzcd}
$$

Predicate logic

Predicate logic

What are functorial semantics for first-order logic?

Predicate logic

What are functorial semantics for first-order logic?
There are different answers.

Predicate logic

What are functorial semantics for first-order logic?
There are different answers. One is Lawvere's "hyperdoctrines".

Predicate logic

What are functorial semantics for first-order logic?
There are different answers. One is Lawvere's "hyperdoctrines".
This involves *fibrations*:

Predicate logic

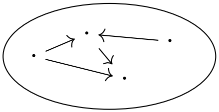
What are functorial semantics for first-order logic?
There are different answers. One is Lawvere's "hyperdoctrines".
This involves *fibrations*:

- "Base category" of contexts and terms
  (finite product category)

Predicate logic

What are functorial semantics for first-order logic?
There are different answers. One is Lawvere's "hyperdoctrines".
This involves *fibrations*:

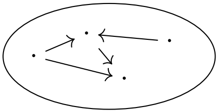
- "Base category" of contexts and terms
  (finite product category)



Predicate logic

What are functorial semantics for first-order logic?
There are different answers. One is Lawvere's "hyperdoctrines".
This involves *fibrations*:

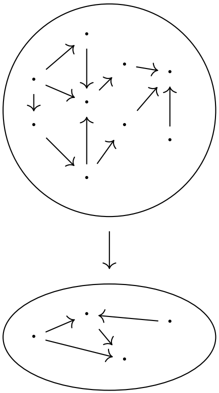
- "Base category" of contexts and terms (finite product category)
- "Total category" of formulas and implications



Predicate logic

What are functorial semantics for first-order logic?
There are different answers. One is Lawvere's "hyperdoctrines".
This involves *fibrations*:

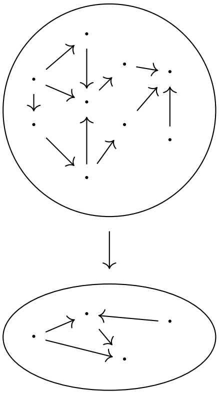
- "Base category" of contexts and terms (finite product category)
- "Total category" of formulas and implications



Predicate logic

What are functorial semantics for first-order logic?
There are different answers. One is Lawvere's "hyperdoctrines".
This involves *fibrations*:

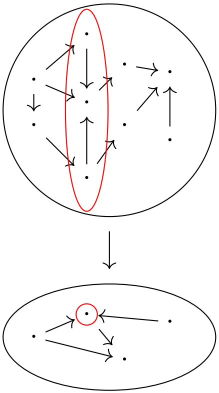
- "Base category" of contexts and terms (finite product category)
- "Total category" of formulas and implications
- "Fibers" are Heyting algebras



Predicate logic

What are functorial semantics for first-order logic?
There are different answers. One is Lawvere's "hyperdoctrines".
This involves *fibrations*:

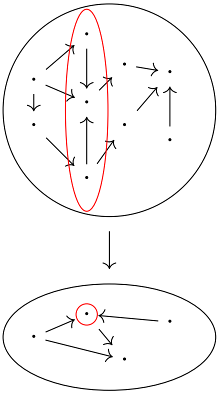
- "Base category" of contexts and terms (finite product category)
- "Total category" of formulas and implications
- "Fibers" are Heyting algebras



Predicate logic

What are functorial semantics for first-order logic?
There are different answers. One is Lawvere's "hyperdoctrines".
This involves *fibrations*:

- "Base category" of contexts and terms (finite product category)
- "Total category" of formulas and implications
- "Fibers" are Heyting algebras
- "Proof-theoretic" version: fibers are non-posetal



Fibrational semantics

Fibrational semantics

Semantics are given by morphisms of fibrations into a "standard fibration".

Fibrational semantics

Semantics are given by morphisms of fibrations into a "standard fibration".

$$\begin{tikzcd} \mathbf{Form}_{\mathcal{L}} \arrow[r] \arrow[d] & \mathbf{Sub}(\mathbf{Set}) \arrow[d] \\ \mathbf{Ctx}_{\mathcal{L}} \arrow[r] & \mathbf{Set} \end{tikzcd}$$

Fibrational semantics

Semantics are given by morphisms of fibrations into a "standard fibration".

$$\begin{array}{ccc} \mathbf{Form}_{\mathcal{L}} & \longrightarrow & \mathbf{Sub}(\mathbf{Set}) \\ \downarrow & & \downarrow \\ \mathbf{Ctx}_{\mathcal{L}} & \longrightarrow & \mathbf{Set} \end{array}$$

"Propositions-as-objects-of-$\mathbf{C}$" semantics are obtained from the non-posetal version.

Fibrational semantics

Semantics are given by morphisms of fibrations into a "standard fibration".

$$
\begin{array}{ccc}
\color{red}{\mathbf{Form}_{\mathcal{L}}} & \longrightarrow & \mathbf{Sub}(\mathbf{Set}) \\
\downarrow & & \downarrow \\
\mathbf{Ctx}_{\mathcal{L}} & \longrightarrow & \mathbf{Set}
\end{array}
$$

"Propositions-as-objects-of-**C**" semantics are obtained from the non-posetal version.

Fibrational semantics

Semantics are given by morphisms of fibrations into a "standard fibration".

$$\begin{array}{ccc} \mathbf{Pf}_{\mathcal{L}} & \longrightarrow & \mathbf{Sub}(\mathbf{Set}) \\ \downarrow & & \downarrow \\ \mathbf{Ctx}_{\mathcal{L}} & \longrightarrow & \mathbf{Set} \end{array}$$

"Propositions-as-objects-of-**C**" semantics are obtained from the non-posetal version.

Fibrational semantics

Semantics are given by morphisms of fibrations into a "standard fibration".

$$\begin{tikzcd} \mathbf{Pf}_{\mathcal{L}} \arrow[r] \arrow[d] & \color{red}{\mathbf{Sub}(\mathbf{Set})} \arrow[d] \\ \mathbf{Ctx}_{\mathcal{L}} \arrow[r] & \color{red}{\mathbf{Set}} \end{tikzcd}$$

"Propositions-as-objects-of-**C**" semantics are obtained from the non-posetal version.

Fibrational semantics

Semantics are given by morphisms of fibrations into a "standard fibration".

$$
\begin{tikzcd}
\mathbf{Pf}_{\mathcal{L}} \arrow[r] \arrow[d] & \mathbf{C}^{\to} \arrow[d, "\mathsf{cod}"] \\
\mathbf{Ctx}_{\mathcal{L}} \arrow[r] & \mathbf{C}
\end{tikzcd}
$$

"Propositions-as-objects-of-$\mathbf{C}$" semantics are obtained from the non-posetal version.

Freeness

Freeness

The fibration $\begin{array}{c}\mathbf{Pf}_{\mathcal{L}}\\ \downarrow\\ \mathbf{Ctx}_{\mathcal{L}}\end{array}$ is free, in two senses.

Freeness

The fibration $\begin{matrix}\mathbf{Pf}_{\mathcal{L}}\\ \downarrow\\ \mathbf{Ctx}_{\mathcal{L}}\end{matrix}$ is free, in two senses. It is the free "Heyting-fibration" *over* $\mathbf{Ctx}_{\mathcal{L}}$

Freeness

The fibration $\begin{array}{c}\mathbf{Pf}_{\mathcal{L}}\\ \downarrow\\ \mathbf{Ctx}_{\mathcal{L}}\end{array}$ is free, in two senses. It is the free "Heyting-fibration" *over* $\mathbf{Ctx}_{\mathcal{L}}$, and $\mathbf{Ctx}_{\mathcal{L}}$ is the free f.p. category with an $\mathcal{L}$-structure.

Freeness

The fibration $\begin{matrix}\mathbf{Pf}_{\mathcal{L}}\\ \downarrow\\ \mathbf{Ctx}_{\mathcal{L}}\end{matrix}$ is free, in two senses. It is the free "Heyting-fibration" *over* $\mathbf{Ctx}_{\mathcal{L}}$, and $\mathbf{Ctx}_{\mathcal{L}}$ is the free f.p. category with an $\mathcal{L}$-structure.

$$\begin{tikzcd}
& \mathbf{Pf}_{\mathcal{L}} \arrow[d] & \mathbf{E} \arrow[d] \\
& \mathbf{Ctx}_{\mathcal{L}} & \mathbf{B} \\
\mathcal{L} \arrow[ur] & &
\end{tikzcd}$$

Freeness

The fibration $\begin{matrix}\mathbf{Pf}_{\mathcal{L}} \\ \downarrow \\ \mathbf{Ctx}_{\mathcal{L}}\end{matrix}$ is free, in two senses. It is the free "Heyting-fibration" *over* $\mathbf{Ctx}_{\mathcal{L}}$, and $\mathbf{Ctx}_{\mathcal{L}}$ is the free f.p. category with an $\mathcal{L}$-structure.

$$\begin{tikzcd}
 & \mathbf{Pf}_{\mathcal{L}} \arrow[d] & \mathbf{E} \arrow[d] \\
 & \mathbf{Ctx}_{\mathcal{L}} & \mathbf{B} \\
\mathcal{L} \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

The fibration $\begin{matrix}\mathbf{Pf}_{\mathcal{L}}\\ \downarrow\\ \mathbf{Ctx}_{\mathcal{L}}\end{matrix}$ is free, in two senses. It is the free "Heyting-fibration" *over* $\mathbf{Ctx}_{\mathcal{L}}$, and $\mathbf{Ctx}_{\mathcal{L}}$ is the free f.p. category with an $\mathcal{L}$-structure.

$$\begin{tikzcd}
 & \mathbf{Pf}_{\mathcal{L}} \arrow[d] & \mathbf{E} \arrow[d] \\
 & \mathbf{Ctx}_{\mathcal{L}} \arrow[r, dashed] & \mathbf{B} \\
\mathcal{L} \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

Freeness

The fibration $\begin{array}{c}\mathbf{Pf}_{\mathcal{L}}\\ \downarrow\\ \mathbf{Ctx}_{\mathcal{L}}\end{array}$ is free, in two senses. It is the free "Heyting-fibration" *over* $\mathbf{Ctx}_{\mathcal{L}}$, and $\mathbf{Ctx}_{\mathcal{L}}$ is the free f.p. category with an $\mathcal{L}$-structure.

$$\begin{tikzcd}
 & \mathbf{Pf}_{\mathcal{L}} \arrow[r, dashed] \arrow[d] & \mathbf{E} \arrow[d] \\
 & \mathbf{Ctx}_{\mathcal{L}} \arrow[r, dashed] & \mathbf{B} \\
\mathcal{L} \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

Invariance

Invariance

Again, we have a “categorical” freeness property

Invariance

Again, we have a "categorical" freeness property, and this gives us isomorphism invariance.

Invariance

Again, we have a "categorical" freeness property, and this gives us isomorphism invariance.

$$
\begin{tikzcd}
 & \mathbf{Pf}_{\mathcal{L}} \arrow[r] \arrow[r, bend left] \arrow[d] & \mathbf{E} \arrow[d] \\
 & \mathbf{Ctx}_{\mathcal{L}} \arrow[r] \arrow[r, bend left] & \mathbf{B} \\
\mathcal{L} \arrow[ur] \arrow[urr] \arrow[urr, bend right] & &
\end{tikzcd}
$$

Invariance

Again, we have a “categorical” freeness property, and this gives us isomorphism invariance.

$$
\begin{tikzcd}
 & \mathbf{Pf}_{\mathcal{L}} \arrow[r, bend left] \arrow[r] \arrow[d] & \mathbf{E} \arrow[d] \\
 & \mathbf{Ctx}_{\mathcal{L}} \arrow[r, bend left] \arrow[r] & \mathbf{B} \\
\mathcal{L} \arrow[ur] \arrow[urr] \arrow[urr, bend right] \arrow[urr, phantom, shift right=3ex, "\cong"] & &
\end{tikzcd}
$$

Invariance

Again, we have a "categorical" freeness property, and this gives us isomorphism invariance.

$$
\begin{tikzcd}
& \mathbf{Pf}_{\mathcal{L}} \arrow[r, bend left] \arrow[r] \arrow[d] & \mathbf{E} \arrow[d] \\
& \mathbf{Ctx}_{\mathcal{L}} \arrow[r, bend left] \arrow[r] \arrow[r, phantom, shift left=1.5ex, "\cong"] & \mathbf{B} \\
\mathcal{L} \arrow[ur] \arrow[urr] \arrow[urr, bend right] \arrow[urr, phantom, shift right=2ex, "\cong"] & &
\end{tikzcd}
$$

Invariance

Again, we have a “categorical” freeness property, and this gives us isomorphism invariance.

$$
\begin{tikzcd}
 & \mathbf{Pf}_{\mathcal{L}} \arrow[r, bend left] \arrow[r] \arrow[r, phantom, shift left=2, "\cong"] \arrow[d] & \mathbf{E} \arrow[d] \\
 & \mathbf{Ctx}_{\mathcal{L}} \arrow[r, bend left] \arrow[r] \arrow[r, phantom, shift left=2, "\cong"] & \mathbf{B} \\
\mathcal{L} \arrow[ur] \arrow[urr] \arrow[urr, bend right] \arrow[urr, phantom, shift right=3, "\cong"] & &
\end{tikzcd}
$$

Homotopical semantics, fibrationally

Homotopical semantics, fibrationally

What is the correct “target fibration” for the homotopical semantics?

Homotopical semantics, fibrationally

What is the correct "target fibration" for the homotopical semantics?

Guess: $\mathrm{cod}\ \begin{array}{c}\mathbf{sSet}^{\to}\\ \downarrow\\ \mathbf{sSet}\end{array}$.

Homotopical semantics, fibrationally

What is the correct "target fibration" for the homotopical semantics?

Guess: $\mathrm{cod} \; \begin{array}{c} \mathbf{sSet}^{\to} \\ \downarrow \\ \mathbf{sSet} \end{array}$. Almost, but interpretation of *equality* is wrong!

Homotopical semantics, fibrationally

What is the correct "target fibration" for the homotopical semantics?

Guess: cod $\begin{array}{c}\mathbf{sSet}^{\to}\\ \downarrow\\ \mathbf{sSet}\end{array}$. Almost, but interpretation of *equality* is wrong!

Equality in a fibration is given by a universal property:

Homotopical semantics, fibrationally

What is the correct "target fibration" for the homotopical semantics?

Guess: cod $\begin{array}{c}\mathbf{sSet}^{\to}\\ \downarrow\\ \mathbf{sSet}\end{array}$. Almost, but interpretation of *equality* is wrong!

Equality in a fibration is given by a universal property:

$$\begin{array}{ccc}\top_A & \longrightarrow & \mathrm{Eq}_A\\ \\ A & \xrightarrow{\Delta_A} & A \times A\end{array}$$

Homotopical semantics, fibrationally

What is the correct "target fibration" for the homotopical semantics?

Guess: $\mathrm{cod} \begin{array}{c}\mathbf{sSet}^{\to}\\ \downarrow\\ \mathbf{sSet}\end{array}$. Almost, but interpretation of *equality* is wrong!

Equality in a fibration is given by a universal property:

$$\begin{tikzcd} & P \\ \top_A \arrow[ur] \arrow[r] & \mathrm{Eq}_A \\ A \arrow[r, "\Delta_A"] & A \times A \end{tikzcd}$$

Homotopical semantics, fibrationally

What is the correct “target fibration” for the homotopical semantics?

Guess: $\mathrm{cod} \begin{array}{c} \mathbf{sSet}^{\rightarrow} \\ \downarrow \\ \mathbf{sSet} \end{array}$. Almost, but interpretation of *equality* is wrong!

Equality in a fibration is given by a universal property:

$$\begin{tikzcd} & P \\ \top_A \arrow[ur] \arrow[r] & \mathrm{Eq}_A \arrow[u, dashed] \\ A \arrow[r, "\Delta_A"] & A \times A \end{tikzcd}$$

Homotopical semantics, fibrationally

What is the correct "target fibration" for the homotopical semantics?

Guess: cod $\begin{array}{c}\mathbf{sSet}^{\to}\\ \downarrow\\ \mathbf{sSet}\end{array}$. Almost, but interpretation of *equality* is wrong!

Equality in a fibration is given by a universal property:

$$\begin{tikzcd} & P \\ \top_A \arrow[ur] \arrow[r] & \mathrm{Eq}_A \arrow[u, dashed] \\ A \arrow[r, "\Delta_A"] & A \times A \end{tikzcd}$$

i.e. $$\frac{\top \implies P(a,a)}{a_1 = a_2 \implies P(a_1, a_2)}$$

Homotopical semantics, fibrationally

What is the correct "target fibration" for the homotopical semantics?

Guess: cod $\begin{array}{c}\mathbf{sSet}^{\to}\\ \downarrow \\ \mathbf{sSet}\end{array}$. Almost, but interpretation of *equality* is wrong!

Equality in a fibration is given by a universal property:

$$\begin{tikzcd} & P \\ \top_A \arrow[ur] \arrow[r] & \mathrm{Eq}_A \arrow[u, dashed] \end{tikzcd} \qquad \text{i.e.} \qquad \frac{\top \implies P(a,a)}{a_1 = a_2 \implies P(a_1, a_2)}$$

$$A \xrightarrow{\Delta_A} A \times A$$

In a codomain fibration cod $\begin{array}{c}\mathbf{C}^{\to}\\ \downarrow \\ \mathbf{C}\end{array}$, this is satisfied by the diagonal $\Delta_A : A \to A \times A$.

Homotopical semantics, fibrationally

The path space has this universal property “up to homotopy”$(*)$.

Homotopical semantics, fibrationally

The path space has this universal property "up to homotopy" (*).

$$
\begin{tikzcd}
 & E \\
X \arrow[ur] \arrow[r, "c"] & X^I \arrow[u, dashed]
\end{tikzcd}
$$

$$X \xrightarrow{\Delta_X} X \times X$$

Homotopical semantics, fibrationally

The path space has this universal property "up to homotopy" (*).

$$\begin{tikzcd} & E \\ X \arrow[ur] \arrow[r, "c"] & X^I \arrow[u, dashed] \end{tikzcd}$$

$$X \xrightarrow{\Delta_X} X \times X$$

Idea:

Homotopical semantics, fibrationally

The path space has this universal property "up to homotopy" (*).

$$\begin{tikzcd} & E \\ X \arrow[ur] \arrow[r, "c"] & X^I \arrow[u, dashed] \end{tikzcd}$$

$$X \xrightarrow{\Delta_X} X \times X$$

Idea: replace cod $\begin{array}{c}\mathbf{sSet}^{\to} \\ \downarrow \\ \mathbf{sSet}\end{array}$ with a fibration whose fibers are the *homotopy categories* of $\mathbf{sSet}/X$.

Homotopical semantics, fibrationally

The path space has this universal property "up to homotopy"(*).

$$\begin{tikzcd} & E \\ X \arrow[ur] \arrow[r, "c"] & X^I \arrow[u, dashed] \end{tikzcd}$$

$$X \xrightarrow{\Delta_X} X \times X$$

Idea: replace cod $\begin{array}{c}\mathbf{sSet}^{\to} \\ \downarrow \\ \mathbf{sSet}\end{array}$ with a fibration whose fibers are the *homotopy categories* of $\mathbf{sSet}/X$. It works!

Homotopical semantics, fibrationally

The path space has this universal property "up to homotopy" (*).

$$\begin{tikzcd} & E \\ X \arrow[ur] \arrow[r, "c"] & X^I \arrow[u, dashed] \end{tikzcd}$$

$$X \xrightarrow{\Delta_X} X \times X$$

Idea: replace cod $\begin{matrix}\mathbf{sSet}^{\to} \\ \downarrow \\ \mathbf{sSet}\end{matrix}$ with a fibration whose fibers are the *homotopy categories* of $\mathbf{sSet}/X$. It works! I.e., it is still a Heyting-fibration,

Homotopical semantics, fibrationally

The path space has this universal property "up to homotopy" (*).

$$\begin{tikzcd} & E \\ X \arrow[ur] \arrow[r, "c"] & X^I \arrow[u, dashed] \end{tikzcd}$$

$$X \xrightarrow{\Delta_X} X \times X$$

Idea: replace cod $\begin{array}{c}\mathbf{sSet}^{\to}\\ \downarrow \\ \mathbf{sSet}\end{array}$ with a fibration whose fibers are the *homotopy categories* of $\mathbf{sSet}/X$. It works! I.e., it is still a Heyting-fibration, with equality given by path spaces.

Homotopical semantics, fibrationally

The path space has this universal property "up to homotopy" (*).

$$
\begin{tikzcd}
 & E \\
X \arrow[ur] \arrow[r, "c"] & X^I \arrow[u, dashed] \\
X \arrow[r, "\Delta_X"] & X \times X
\end{tikzcd}
$$

Idea: replace cod $\begin{array}{c}\mathbf{sSet}^{\to}\\ \downarrow\\ \mathbf{sSet}\end{array}$ with a fibration whose fibers are the *homotopy categories* of $\mathbf{sSet}/X$. It works! I.e., it is still a Heyting-fibration, with equality given by path spaces. (In fact, this works with **Top** as well!)

Homotopy invariance

Homotopy invariance

How can we express homotopy invariance with this setup?

Homotopy invariance

How can we express homotopy invariance with this setup?

$$\begin{tikzcd}
\mathbf{Pf}_{\mathcal{L}} \arrow[d] & & \mathbf{Ho}(\mathbf{sSet}^{\rightarrow}) \arrow[d] \\
\mathbf{Ctx}_{\mathcal{L}} \arrow[rr, bend left, "M"] \arrow[rr, bend right, "N"'] & & \mathbf{sSet}
\end{tikzcd}$$

Homotopy invariance

How can we express homotopy invariance with this setup?

$$\begin{tikzcd}
\mathbf{Pf}_{\mathcal{L}} \arrow[r, bend left, "\tilde{M}"] \arrow[r, bend right, "\tilde{N}"'] \arrow[d] & \mathbf{Ho}(\mathbf{sSet}^{\rightarrow}) \arrow[d] \\
\mathbf{Ctx}_{\mathcal{L}} \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] & \mathbf{sSet}
\end{tikzcd}$$

Homotopy invariance

How can we express homotopy invariance with this setup?

$$
\begin{tikzcd}
\mathbf{Pf}_{\mathcal{L}} \arrow[r, bend left, "\tilde{M}"] \arrow[r, bend right, "\tilde{N}"'] \arrow[d] & \mathbf{Ho}(\mathbf{sSet}^{\to}) \arrow[d] \\
\mathbf{Ctx}_{\mathcal{L}} \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "?\Downarrow"] & \mathbf{sSet}
\end{tikzcd}
$$

Homotopy invariance

How can we express homotopy invariance with this setup?

$$\begin{tikzcd}
\mathbf{Pf}_{\mathcal{L}} \arrow[r, bend left, "\tilde{M}"] \arrow[r, bend right, "\tilde{N}"'] \arrow[r, phantom, "?\Downarrow"] \arrow[d] & \mathbf{Ho}(\mathbf{sSet}^{\to}) \arrow[d] \\
\mathbf{Ctx}_{\mathcal{L}} \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "?\Downarrow"] & \mathbf{sSet}
\end{tikzcd}$$

Homotopy invariance

How can we express homotopy invariance with this setup?

$$\begin{tikzcd}
\mathbf{Pf}_{\mathcal{L}} \arrow[r, bend left, "\tilde{M}"] \arrow[r, bend right, "\tilde{N}"'] \arrow[r, phantom, "?\downarrow"] \arrow[d] & \mathbf{Ho}(\mathbf{sSet}^{\rightarrow}) \arrow[d] \\
\mathbf{Ctx}_{\mathcal{L}} \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "?\downarrow"] & \mathbf{sSet}
\end{tikzcd}$$

(Partial) answer:

Homotopy invariance

How can we express homotopy invariance with this setup?

$$\begin{tikzcd} \mathbf{Pf}_{\mathcal{L}} \arrow[r, bend left, "\tilde{M}"] \arrow[r, bend right, "\tilde{N}"'] \arrow[r, phantom, "?\downarrow"] \arrow[d] & \mathbf{Ho}(\mathbf{sSet}^{\rightarrow}) \arrow[d] \\ \mathbf{Ctx}_{\mathcal{L}} \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "?\downarrow"] & \mathbf{sSet} \end{tikzcd}$$

(Partial) answer: $M \xrightarrow{?} N$ is a *pseudo-natural transformation* into the *homotopy 2-category* of simplicial sets.

Homotopy invariance

How can we express homotopy invariance with this setup?

$$
\begin{tikzcd}
\mathbf{Pf}_{\mathcal{L}} \arrow[r, bend left, "\tilde{M}"] \arrow[r, bend right, "\tilde{N}"'] \arrow[r, phantom, "?\downarrow"] \arrow[d] & \mathbf{Ho}(\mathbf{sSet}^{\rightarrow}) \arrow[d] \\
\mathbf{Ctx}_{\mathcal{L}} \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "?\downarrow"] & \mathbf{sSet}
\end{tikzcd}
$$

(Partial) answer: $M \overset{?}{\to} N$ is a *pseudo-natural transformation* into the *homotopy 2-category* of simplicial sets.

$$
\begin{tikzcd}
M(A) \arrow[r, "h_A"] \arrow[d, "M(f)"'] \arrow[dr, phantom, "\overset{\simeq}{\Lleftarrow}"] & N(A) \arrow[d, "N(f)"] \\
M(B) \arrow[r, "h_B"] & N(B)
\end{tikzcd}
$$

1-discrete 2-fibrations

1-discrete 2-fibrations

Fibrations arise as pullbacks

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

$$\begin{tikzcd} \mathbf{C} \arrow[d, "\mathcal{C}"'] \\ \mathbf{B} \end{tikzcd}$$

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

$$
\begin{array}{ccc}
\mathbf{C} & & \mathbf{Cat}_*^{\mathrm{op}} \\
{\scriptstyle \mathcal{C}}\downarrow & & \downarrow \\
\mathbf{B} & & \mathbf{Cat}^{\mathrm{op}}
\end{array}
$$

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

$$
\begin{tikzcd}
\mathbf{C} \arrow[r] \arrow[d, "\mathcal{C}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_*^{\mathrm{op}} \arrow[d] \\
\mathbf{B} \arrow[r, "\widehat{\mathcal{C}}"] & \mathbf{Cat}^{\mathrm{op}}
\end{tikzcd}
$$

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

$$\begin{tikzcd}
\mathbf{C} \arrow[r] \arrow[d, "\mathcal{C}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_*^{\mathrm{op}} \arrow[d] \\
\mathbf{B} \arrow[r, "\widehat{\mathcal{C}}"] & \mathbf{Cat}^{\mathrm{op}}
\end{tikzcd}$$

Here, $\widehat{\mathcal{C}}$ is a *pseudofunctor*

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

$$\begin{tikzcd}
\mathbf{C} \arrow[r] \arrow[d, "\mathcal{C}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_*^{\mathrm{op}} \arrow[d] \\
\mathbf{B} \arrow[r, "\widehat{\mathcal{C}}"] & \mathbf{Cat}^{\mathrm{op}}
\end{tikzcd}$$

Here, $\widehat{\mathcal{C}}$ is a *pseudofunctor* and $\mathbf{Cat}$ is considered as a *2-category*.

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

$$\begin{tikzcd}
\mathbf{C} \arrow[r] \arrow[d, "\mathcal{C}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_*^{\mathrm{op}} \arrow[d] \\
\mathbf{B} \arrow[r, "\widehat{\mathcal{C}}"] & \mathbf{Cat}^{\mathrm{op}}
\end{tikzcd}$$

Here, $\widehat{\mathcal{C}}$ is a *pseudofunctor* and **Cat** is considered as a *2-category*.
This still makes sense

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

$$\begin{tikzcd}
\mathbf{C} \arrow[r] \arrow[d, "\mathcal{C}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_*^{\mathrm{op}} \arrow[d] \\
\mathbf{B} \arrow[r, "\widehat{\mathcal{C}}"] & \mathbf{Cat}^{\mathrm{op}}
\end{tikzcd}$$

Here, $\widehat{\mathcal{C}}$ is a *pseudofunctor* and **Cat** is considered as a *2-category*.
This still makes sense when **B** is also a 2-category.

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

$$\begin{tikzcd}
\mathbf{C} \arrow[r] \arrow[d, "\mathcal{C}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_*^{\mathrm{op}} \arrow[d] \\
\mathbf{B} \arrow[r, "\widehat{\mathcal{C}}"] & \mathbf{Cat}^{\mathrm{op}}
\end{tikzcd}$$

Here, $\widehat{\mathcal{C}}$ is a *pseudofunctor* and **Cat** is considered as a *2-category*.
This still makes sense when **B** is also a 2-category.
The resulting notion

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

$$\begin{tikzcd} \mathbf{C} \arrow[r] \arrow[d, "\mathcal{C}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_*^{\mathrm{op}} \arrow[d] \\ \mathbf{B} \arrow[r, "\widehat{\mathcal{C}}"] & \mathbf{Cat}^{\mathrm{op}} \end{tikzcd}$$

Here, $\widehat{\mathcal{C}}$ is a *pseudofunctor* and **Cat** is considered as a *2-category*.
This still makes sense when **B** is also a 2-category.
The resulting notion is that of a *1-discrete 2-fibration*

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

$$\begin{tikzcd}
\mathbf{C} \arrow[r] \arrow[d, "\mathcal{C}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_*^{\mathrm{op}} \arrow[d] \\
\mathbf{B} \arrow[r, "\widehat{\mathcal{C}}"] & \mathbf{Cat}^{\mathrm{op}}
\end{tikzcd}$$

Here, $\widehat{\mathcal{C}}$ is a *pseudofunctor* and **Cat** is considered as a *2-category*.
This still makes sense when **B** is also a 2-category.
The resulting notion is that of a *1-discrete 2-fibration*, in which **C** is (also) also a 2-category.

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

$$\begin{tikzcd}
\mathbf{C} \arrow[r] \arrow[d, "\mathcal{C}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_*^{\mathrm{op}} \arrow[d] \\
\mathbf{B} \arrow[r, "\widehat{\mathcal{C}}"] & \mathbf{Cat}^{\mathrm{op}}
\end{tikzcd}$$

Here, $\widehat{\mathcal{C}}$ is a *pseudofunctor* and $\mathbf{Cat}$ is considered as a *2-category*. This still makes sense when $\mathbf{B}$ is also a 2-category.
The resulting notion is that of a *1-discrete 2-fibration*, in which $\mathbf{C}$ is (also) also a 2-category.

The pseudo-functor $\mathbf{sSet} \to \mathbf{Cat}^{\mathrm{op}}$

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

$$\begin{tikzcd} \mathbf{C} \arrow[r] \arrow[d, "\mathcal{C}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_*^{\mathrm{op}} \arrow[d] \\ \mathbf{B} \arrow[r, "\widehat{\mathcal{C}}"] & \mathbf{Cat}^{\mathrm{op}} \end{tikzcd}$$

Here, $\widehat{\mathcal{C}}$ is a *pseudofunctor* and **Cat** is considered as a *2-category*.
This still makes sense when **B** is also a 2-category.
The resulting notion is that of a *1-discrete 2-fibration*, in which **C** is (also) also a 2-category.

The pseudo-functor $\mathbf{sSet} \to \mathbf{Cat}^{\mathrm{op}}$ associated to $\begin{array}{c} \mathbf{Ho}(\mathbf{sSet}^{\to}) \\ \downarrow \\ \mathbf{sSet} \end{array}$

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

$$\begin{tikzcd}
\mathbf{C} \arrow[r] \arrow[d, "\mathcal{C}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_*^{\mathrm{op}} \arrow[d] \\
\mathbf{B} \arrow[r, "\widehat{\mathcal{C}}"] & \mathbf{Cat}^{\mathrm{op}}
\end{tikzcd}$$

Here, $\widehat{\mathcal{C}}$ is a *pseudofunctor* and $\mathbf{Cat}$ is considered as a *2-category*.
This still makes sense when $\mathbf{B}$ is also a 2-category.
The resulting notion is that of a *1-discrete 2-fibration*, in which $\mathbf{C}$ is (also) also a 2-category.

The pseudo-functor $\mathbf{sSet} \to \mathbf{Cat}^{\mathrm{op}}$ associated to $\begin{array}{c}\mathbf{Ho}(\mathbf{sSet}^{\to}) \\ \downarrow \\ \mathbf{sSet}\end{array}$ extends to the 2-category $\mathbf{sSet}$

1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\mathrm{op}}$.

$$\begin{tikzcd} \mathbf{C} \arrow[r] \arrow[d, "\mathcal{C}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_*^{\mathrm{op}} \arrow[d] \\ \mathbf{B} \arrow[r, "\widehat{\mathcal{C}}"] & \mathbf{Cat}^{\mathrm{op}} \end{tikzcd}$$

Here, $\widehat{\mathcal{C}}$ is a *pseudofunctor* and **Cat** is considered as a *2-category*.
This still makes sense when **B** is also a 2-category.
The resulting notion is that of a *1-discrete 2-fibration*, in which **C** is (also) also a 2-category.

The pseudo-functor $\mathbf{sSet} \to \mathbf{Cat}^{\mathrm{op}}$ associated to $\begin{array}{c}\mathbf{Ho}(\mathbf{sSet}^{\to})\\ \downarrow \\ \mathbf{sSet}\end{array}$ extends to the 2-category **sSet**, hence this fibration extends to a 1D2F.

Homotopy invariance

Homotopy invariance

The desired homotopy-invariance property

Homotopy invariance

The desired homotopy-invariance property then amounts to the existence of a pseudo-natural equivalence $\tilde{\alpha}$

Homotopy invariance

The desired homotopy-invariance property then amounts to the existence of a pseudo-natural equivalence $\tilde{\alpha}$

$$\begin{tikzcd}
\mathbf{Pf}_{\mathcal{L}} \arrow[r, bend left, "\tilde{M}"] \arrow[r, bend right, "\tilde{N}"'] \arrow[d] & \mathbf{Ho}(\mathbf{sSet}^{\rightarrow}) \arrow[d] \\
\mathbf{Ctx}_{\mathcal{L}} \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] & \mathbf{sSet}
\end{tikzcd}$$

Homotopy invariance

The desired homotopy-invariance property then amounts to the existence of a pseudo-natural equivalence $\tilde{\alpha}$

$$\begin{tikzcd}
\mathbf{Pf}_{\mathcal{L}} \arrow[rr, bend left, "\tilde{M}"] \arrow[rr, bend right, "\tilde{N}"'] \arrow[d] & & \mathbf{Ho}(\mathbf{sSet}^{\rightarrow}) \arrow[d] \\
\mathbf{Ctx}_{\mathcal{L}} \arrow[rr, bend left, "M"] \arrow[rr, bend right, "N"'] \arrow[rr, phantom, "\alpha \Downarrow \simeq"] & & \mathbf{sSet}
\end{tikzcd}$$

Homotopy invariance

The desired homotopy-invariance property then amounts to the existence of a pseudo-natural equivalence $\tilde{\alpha}$

$$
\begin{tikzcd}
\mathbf{Pf}_{\mathcal{L}} \arrow[r, bend left, "\tilde{M}"] \arrow[r, bend right, "\tilde{N}"'] \arrow[r, phantom, "\tilde{\alpha}\Downarrow\simeq"] \arrow[d] & \mathbf{Ho}(\mathbf{sSet}^{\to}) \arrow[d] \\
\mathbf{Ctx}_{\mathcal{L}} \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\alpha\Downarrow\simeq"] & \mathbf{sSet}
\end{tikzcd}
$$

Homotopy invariance

The desired homotopy-invariance property then amounts to the existence of a pseudo-natural equivalence $\tilde{\alpha}$ over a given pseudo-natural equivalence α.

$$\begin{tikzcd}
\mathbf{Pf}_{\mathcal{L}} \arrow[r, bend left, "\tilde{M}"] \arrow[r, bend right, "\tilde{N}"'] \arrow[r, phantom, "\tilde{\alpha}\downarrow\simeq"] \arrow[d] & \mathbf{Ho}(\mathbf{sSet}^{\to}) \arrow[d] \\
\mathbf{Ctx}_{\mathcal{L}} \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\alpha\downarrow\simeq"] & \mathbf{sSet}
\end{tikzcd}$$

Homotopy invariance

The desired homotopy-invariance property then amounts to the existence of a pseudo-natural equivalence $\tilde{\alpha}$ over a given pseudo-natural equivalence α.

$$\begin{tikzcd}
\mathbf{Pf}_{\mathcal{L}} \arrow[r, bend left, "\tilde{M}"] \arrow[r, bend right, "\tilde{N}"'] \arrow[r, phantom, "\tilde{\alpha}\Downarrow\simeq"] \arrow[d] & \mathbf{Ho}(\mathbf{sSet}^{\to}) \arrow[d] \\
\mathbf{Ctx}_{\mathcal{L}} \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\alpha\Downarrow\simeq"] & \mathbf{sSet}
\end{tikzcd}$$

This can again be shown from the freeness property of $\begin{array}{c}\mathbf{Pf}_{\mathcal{L}} \\ \downarrow \\ \mathbf{Ctx}_{\mathcal{L}}\end{array}$.

Outline

The abstract invariance theorem

This argument depended heavily on the special nature of the category **sSet**.

The abstract invariance theorem

This argument depended heavily on the special nature of the category **sSet**. (The *isomorphism* invariance property

The abstract invariance theorem

This argument depended heavily on the special nature of the category **sSet**. (The *isomorphism* invariance property, by contrast,

The abstract invariance theorem

This argument depended heavily on the special nature of the category **sSet**. (The *isomorphism* invariance property, by contrast, does not.)

The abstract invariance theorem

This argument depended heavily on the special nature of the category **sSet**. (The *isomorphism* invariance property, by contrast, does not.) To put the proof in the proper, general context, we should

The abstract invariance theorem

This argument depended heavily on the special nature of the category **sSet**. (The *isomorphism* invariance property, by contrast, does not.) To put the proof in the proper, general context, we should

- Show that for *any* Heyting fibration $\begin{array}{c}\mathbf{E}\\ \downarrow\\ \mathbf{B}\end{array}$, there is natural 2-categorical structure on $\mathbf{B}$

The abstract invariance theorem

This argument depended heavily on the special nature of the category **sSet**. (The *isomorphism* invariance property, by contrast, does not.) To put the proof in the proper, general context, we should

- Show that for *any* Heyting fibration $\begin{array}{c}\mathbf{E}\\ \downarrow\\ \mathbf{B}\end{array}$, there is natural 2-categorical structure on $\mathbf{B}$
- (and that this recovers the usual one on **sSet** and **Top**)

The abstract invariance theorem

This argument depended heavily on the special nature of the category **sSet**. (The *isomorphism* invariance property, by contrast, does not.) To put the proof in the proper, general context, we should

- Show that for *any* Heyting fibration $\begin{matrix}\mathbf{E}\\ \downarrow\\ \mathbf{B}\end{matrix}$, there is natural 2-categorical structure on $\mathbf{B}$
- (and that this recovers the usual one on **sSet** and **Top**)
- Show that the associated pseudofunctor $\mathbf{B} \to \mathbf{Cat}^{\mathrm{op}}$ is a pseudo-functor of 2-categories

The abstract invariance theorem

This argument depended heavily on the special nature of the category **sSet**. (The *isomorphism* invariance property, by contrast, does not.) To put the proof in the proper, general context, we should

- Show that for *any* Heyting fibration $\begin{array}{c}\mathbf{E}\\ \downarrow\\ \mathbf{B}\end{array}$, there is natural 2-categorical structure on $\mathbf{B}$
- (and that this recovers the usual one on **sSet** and **Top**)
- Show that the associated pseudofunctor $\mathbf{B} \to \mathbf{Cat}^{\mathrm{op}}$ is a pseudo-functor of 2-categories (thus giving us a 1D2F)

The abstract invariance theorem

This argument depended heavily on the special nature of the category **sSet**. (The *isomorphism* invariance property, by contrast, does not.) To put the proof in the proper, general context, we should

- Show that for *any* Heyting fibration $\begin{array}{c}\mathbf{E}\\ \downarrow\\ \mathbf{B}\end{array}$, there is natural 2-categorical structure on $\mathbf{B}$
- (and that this recovers the usual one on **sSet** and **Top**)
- Show that the associated pseudofunctor $\mathbf{B} \to \mathbf{Cat}^{\mathrm{op}}$ is a pseudo-functor of 2-categories (thus giving us a 1D2F)
- We can do it!

The abstract invariance theorem

This argument depended heavily on the special nature of the category **sSet**. (The *isomorphism* invariance property, by contrast, does not.) To put the proof in the proper, general context, we should

- Show that for *any* Heyting fibration $\begin{array}{c}\mathbf{E}\\ \downarrow\\ \mathbf{B}\end{array}$, there is natural 2-categorical structure on **B**
- (and that this recovers the usual one on **sSet** and **Top**)
- Show that the associated pseudofunctor $\mathbf{B} \to \mathbf{Cat}^{\mathrm{op}}$ is a pseudo-functor of 2-categories (thus giving us a 1D2F)
- We can do it! In fact, we need much less than a Heyting fibration

The abstract invariance theorem

This argument depended heavily on the special nature of the category **sSet**. (The *isomorphism* invariance property, by contrast, does not.) To put the proof in the proper, general context, we should

- Show that for *any* Heyting fibration $\begin{smallmatrix}\mathbf{E}\\ \downarrow\\ \mathbf{B}\end{smallmatrix}$, there is natural 2-categorical structure on $\mathbf{B}$
- (and that this recovers the usual one on **sSet** and **Top**)
- Show that the associated pseudofunctor $\mathbf{B} \to \mathbf{Cat}^{\mathrm{op}}$ is a pseudo-functor of 2-categories (thus giving us a 1D2F)
- We can do it! In fact, we need much less than a Heyting fibration (a "$\wedge$=-fibration" is good enough)

The abstract invariance theorem

This argument depended heavily on the special nature of the category **sSet**. (The *isomorphism* invariance property, by contrast, does not.) To put the proof in the proper, general context, we should

- Show that for *any* Heyting fibration $\begin{array}{c}\mathbf{E}\\ \downarrow\\ \mathbf{B}\end{array}$, there is natural 2-categorical structure on **B**
- (and that this recovers the usual one on **sSet** and **Top**)
- Show that the associated pseudofunctor $\mathbf{B} \to \mathbf{Cat}^{\mathrm{op}}$ is a pseudo-functor of 2-categories (thus giving us a 1D2F)
- We can do it! In fact, we need much less than a Heyting fibration (a "$\wedge$=-fibration" is good enough)
- The 2-categorical structure on **B** is given by the "internal" notion of homotopy/equality

Thank you for your attention!

For more information, see:

- Homotopies in Grothendieck fibrations (arXiv:1905.10690)
- First-order homotopical logic (forthcoming)