First-Order Homotopical Logic

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1. A diagram
2. Propositions as spaces
3. Properties
4. Fibrational semantics
5. The abstract invariance theorem
Outline

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2. Propositions as spaces
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Consider the diagram...

MLTT \[\xrightarrow{\text{Voevodsky-Awodey-Warren-Kapulkin-Lumsdaine-\cdots}}\] Simplicial Sets

IFOL

Martin-Löf 1972
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**First-order homotopical logic**
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Propositions as objects of $\mathcal{C}$
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Fix a first-order language \( \mathcal{L} \)
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The “Propositions-as-Objects-of-$\mathbf{C}$” semantics should assign to each formula $\phi$ with free variables in $\Gamma$ an object $M_\Gamma(\phi) \in \text{Ob}(\mathbf{C}/M(\Gamma))$ (a “family of objects” over $M(\Gamma)$).
Propositions as objects of $\mathbf{C}$

Possible if $\mathbf{C}$ is locally cartesian closed and has finite coproducts.
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<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$M_\Gamma(\phi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P \land Q$</td>
<td>$M_\Gamma(P) \times M_\Gamma(Q)$</td>
</tr>
<tr>
<td>$P \lor Q$</td>
<td>$M_\Gamma(P) + M_\Gamma(Q)$</td>
</tr>
<tr>
<td>$P \Rightarrow Q$</td>
<td>$M_\Gamma(Q)^{M_\Gamma(P)}$</td>
</tr>
<tr>
<td>$(\forall x \in A)P$</td>
<td>$\prod_{\pi : M(\Gamma \cup {x}) \to M(\Gamma)} M_\Gamma(P)$</td>
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<td>$(\exists x \in A)P$</td>
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$M(x, y)(x = y)$

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Propositions as spaces

Now take $\mathcal{C} =$ “spaces” (e.g., simplicial sets, which is LCCC).
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But

Equality should be "paths", not "identity"

Instead of $\Delta : X \to X \times X$, use the "path-space fibration" $X^I \to X \times X$.

End of definition of the homotopical semantics for first-order logic.

For homotopical semantics in topological spaces, start with an $L$-structure in $\text{Top}$ and apply $\text{Sing} : \text{Top} \to \text{sSet}$.

Let us say $M \models \phi$ if $M(\phi) \neq \emptyset$. 
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- Instead of $\Delta^1 : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$, use the “path-space fibration” $\mathcal{X} \times I \to \mathcal{X} \times \mathcal{X}$.

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Examples

Let $L = (A, B, f : A \to B, g : A \to B)$.

Then $M = (X, Y, f, g)$ satisfies $\forall x \in A [f(x) = g(x)]$ if and only if $f$ and $g$ are homotopic.

Let $L = (A, \circ : A \times A \to A)$.

Then $M = (M, \circ)$ satisfies $\forall x, y, z [x \circ (y \circ z) = (x \circ y) \circ z]$ if and only if $\circ$ is homotopy-associative.

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Let $\mathcal{L} = (A, B, f : A \rightarrow B, g : A \rightarrow B)$. Then $M = (X, Y, f, g)$ satisfies $\forall x \in A[f(x) = g(x)]$ if and only if $f$ and $g$ are homotopic.

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Properties

Some properties one might expect/hope for:
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- Soundness

Consider $\exists x \forall y \left( \neg \neg x = y \right) \Rightarrow \exists x \forall y \left( x = y \right)$. A space $X$ satisfies

- the antecedent if and only if $X$ is path-connected.
- the consequent if and only if $X$ is contractible.
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  - For intuitionistic propositional logic

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    - the antecedent if and only if $X$ is path-connected.
    - the consequent if and only if $X$ is contractible.

- **Completeness**
  - ???

- **Homotopy-invariance**
Invariance

The classical (Tarskian) semantics is isomorphism-invariant. I.e., for isomorphic \( L \)-structures, \( M \) and \( N \) and a sentence \( \phi \):

- \( M \models \phi \) if and only if \( N \models \phi \) (and an analogous property for non-closed formulas).

Easy proof by induction on \( \phi \).

For Propositions-as-objects-of-C, this can be strengthened:

- For \( M \), \( N \), and \( \phi \) as above, \( M(\phi) \) and \( N(\phi) \) are (“canonically”) isomorphic (and again something for non-closed formulas).

Again, easy inductive proof.
Invariance

The **classical (Tarskian) semantics** is *isomorphism-invariant.*
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• I.e., for isomorphic \( \mathcal{L} \)-structures, \( M \) and \( N \) and a sentence \( \phi \):

\[
M \vDash \phi \quad \text{if and only if} \quad N \vDash \phi
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For **Propositions-as-objects-of-C**, this can be strengthened:

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(and again something for non-closed formulas).

• Again, easy inductive proof
Homotopy-invariance

The homotopical semantics satisfy an even stronger(*) property:

- Given homotopy-equivalent structures $M$ and $N$,
  $M(\varphi)$ and $N(\varphi)$ are homotopy-equivalent.

- Here, $M$ and $N$ are homotopy equivalent if there are homotopy-equivalences $h_A : M(A) \cong N(A)$ s.t.
  $M(A) \times M(B) \cong N(A) \times N(B)$,
  $M(C) \cong N(C)$,
  $h_C$ etc. commute up to homotopy.

- Can be proven by induction, but not so easily.

- There is a more conceptual (and general) proof using "fibrational" semantics.
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  \[
  \begin{array}{cccc}
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  M(f) \downarrow & & & \downarrow N(f) \\
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Outline

1 A diagram

2 Propositions as spaces

3 Properties

4 Fibrational semantics

5 The abstract invariance theorem
Functorial semantics (Lawvere)
Functorial semantics (Lawvere)

Syntax
Functorial semantics (Lawvere)
Functorial semantics (Lawvere)

Syntax \rightarrow \text{Semantics} \rightarrow \text{The Universe}
Functorial semantics (Lawvere)

Syntax \xrightarrow{\text{Semantics}} \text{The Universe}
Functorial semantics (Lawvere)

Some category \rightarrow \text{Semantics} \rightarrow \text{The Universe}
Functorial semantics (Lawvere)

Some category \rightarrow \text{Semantics} \rightarrow \text{The Universe}
Functorial semantics (Lawvere)

Some category

Semantics

Some other category (probably Set)
Functorial semantics (Lawvere)

Some category \rightarrow \text{Semantics} \rightarrow \text{Some other category (probably } Set)
Functorial semantics (Lawvere)

Some category \(\rightarrow\) Some functor \(\rightarrow\) Some other category (probably \(\text{Set}\))
Functorial semantics (Lawvere)

Some category \rightarrow Some functor \rightarrow Some other category (probably Set)
Functorial semantics (Lawvere)

Some Boolean (or Heyting) algebra

Some functor

2
Freeness

The Boolean/Heyting algebra $B\Sigma$ of propositions over a set $\Sigma$ of atoms is free. Instead, can take the free "non-posetal Heyting algebra" (CCC w/ finite coproducts) $C\Sigma$. This is Lambek’s "category of proofs". This gives the "Propositions-as-objects-of $C\Sigma$" semantics.
Freeness

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The Boolean/Heyting algebra $B_{\Sigma}$ of propositions over a set $\Sigma$ of atoms is \textit{free}.

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\begin{array}{c}
B_{\Sigma} \\
\uparrow \\
\Sigma
\end{array}
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\[ B_{\Sigma} \rightarrow 2 \]

\[ \Sigma \]
The Boolean/Heyting algebra $B_\Sigma$ of propositions over a set $\Sigma$ of atoms is \textit{free}.

\[
\begin{array}{ccc}
B_\Sigma & \longrightarrow & 2 \\
\uparrow & & \downarrow \\
\Sigma & \longrightarrow & \\
\end{array}
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\[
B_\Sigma \longrightarrow 2
\]

\[
\Sigma
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Invariance

Using the "categorical" universal property of $C$, we obtain an "isomorphism invariance" property: $C\sigma Set \Sigma$.
Using the “categorical” universal property of $\mathbf{C}_\Sigma$
Invariance

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![Diagram](image)
Invariance

Using the “categorical” universal property of $C_\Sigma$, we obtain an “isomorphism invariance” property:

$$C_\sigma \cong \Sigma \cong \text{Set}$$
Predicate logic

What are functorial semantics for first-order logic?

There are different answers. One is Lawvere's "hyperdoctrines". This involves:

- "Base category" of contexts and terms (finite product category)
- "Total category" of formulas and implications
- "Fibers" are Heyting algebras
- "Proof-theoretic" version: fibers are non-posetal
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Fibrational semantics

Semantics are given by morphisms of fibrations into a "standard fibration".

Form

$L$ Sub $(Set)$

"Propositions-as-objects-of-$C$" semantics are obtained from the non-posetal version.
Fibrational semantics

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\[
\begin{align*}
\text{Form}_\mathcal{L} & \longrightarrow \text{Sub}(\text{Set}) \\
\downarrow & \quad \quad \quad \quad \downarrow \\
\text{Ctx}_\mathcal{L} & \longrightarrow \text{Set}
\end{align*}
\]
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\downarrow & & \downarrow \\
\text{Ctx}_L & \rightarrow & \text{Set}
\end{array}
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\[
\begin{array}{ccc}
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\downarrow & & \downarrow \\
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\end{array}
\]

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\]

“Propositions-as-objects-of-\(\mathcal{C}\)” semantics are obtained from the non-posetal version.
Fibrational semantics

Semantics are given by morphisms of fibrations into a “standard fibration”.

\[
\begin{array}{c}
Pf_L \\ \downarrow \\ Ctx_L \\
\end{array} \longrightarrow \begin{array}{c}
C \\ \downarrow \text{cod} \\
C \\
\end{array}
\]

“Propositions-as-objects-of-\(C\)” semantics are obtained from the non-posetal version.
Freeness

The fibration $Pf_L \downarrow Ctx_L$ is free, in two senses. It is the free "Heyting-fibration" over $Ctx_L$, and $Ctx_L$ is the free f.p. category with an $L$-structure.
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\[
\begin{array}{ccc}
\text{Pf}_\mathcal{L} & \downarrow & \text{E} \\
\downarrow & \text{Ctx}_\mathcal{L} & \downarrow \\
\mathcal{L} & \rightarrow & \text{B}
\end{array}
\]
Freeness

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The fibration \( \text{Pf}_L \downarrow \) is free, in two senses. It is the free "Heyting-fibration" over \( \text{Ctx}_L \), and \( \text{Ctx}_L \) is the free f.p. category with an \( L \)-structure.
Invariance

Again, we have a "categorical" freeness property, and this gives us isomorphism invariance.
Invariance

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Homotopical semantics, fibrationally
Homotopical semantics, fibrationally

What is the correct “target fibration” for the homotopical semantics?

Equality in a fibration is given by a universal property:

\[ \top \overset{\Delta}{\to} A \times A \]

In a codomain fibration cod\[C \to \triangleleft C\], this is satisfied by the diagonal \[\Delta A : A \to A \times A\].
Homotopical semantics, fibrationally

What is the correct “target fibration” for the homotopical semantics?

\[ \text{sSet} \to \text{cod} \]

Guess: \( \text{cod} \downarrow \text{sSet} \).

Almost, but interpretation of equality is wrong!

Equality in a fibration is given by a universal property:

\[ \top_{A} \xRightarrow{\phantom{P}} P(a, a) \]

In a codomain fibration \( \text{cod} \to \downarrow \text{C} \), this is satisfied by the diagonal \( \Delta_{A} : A \to A \times A \).
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\[
\text{cod} : \text{sSet} \rightarrow \downarrow_{\text{sSet}} \text{sSet}
\]

Guess: cod \[ \downarrow \]. Almost, but interpretation of \textit{equality} is wrong!
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Equality in a fibration is given by a universal property:

\[
\begin{array}{ccc}
\top_A & \longrightarrow & \text{Eq}_A \\
A & \xrightarrow{\Delta_A} & A \times A
\end{array}
\]
Homotopical semantics, fibrationally

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\[
P \xrightarrow{\top} \text{Eq}_A \xrightarrow{T_A} A \rightarrow A \times A
\]

\[
A \xrightarrow{\Delta_A} A \times A
\]
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\[
\begin{align*}
P & \to \Delta_A \\
\top_A & \to \text{Eq}_A \\
A & \xrightarrow{\Delta_A} A \times A
\end{align*}
\]
Homotopical semantics, fibrationally

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\[
\begin{array}{rcl}
\top_A & \rightarrow & 
\text{Eq}_A \\
\downarrow & & \\
A & \xrightarrow{\Delta_A} & A \times A
\end{array}
\]

i.e. \[
\frac{\top \implies P(a, a)}{a_1 = a_2 \implies P(a_1, a_2)}
\]
Homotopical semantics, fibrationally

What is the correct “target fibration” for the homotopical semantics?

$$\text{Guess: } \text{cod}_{\text{sSet}} \rightarrow \text{sSet}.$$ Almost, but interpretation of equality is wrong!

Equality in a fibration is given by a universal property:

$$\top_A \rightarrow \text{Eq}_A$$

$$A \xrightarrow{\Delta_A} A \times A$$

$$P$$

\[ \frac{\top \Rightarrow P(a, a)}{a_1 = a_2 \Rightarrow P(a_1, a_2)} \]

In a codomain fibration $$\text{cod}_{\text{C}} \rightarrow \text{C}$$, this is satisfied by the diagonal $$\Delta_A : A \rightarrow A \times A.$$
Homotopical semantics, fibrationally

The path space has this universal property “up to homotopy” (\(^*\)).
Homotopical semantics, fibrationally

The path space has this universal property “up to homotopy” (*)

\[
\begin{array}{ccc}
X & \xrightarrow{c} & X^I \\
\downarrow & \text{c} & \downarrow \\
X & \xrightarrow{\Delta_X} & X \times X
\end{array}
\]

\[E \xrightarrow{\text{id}} E \]

Idea: replace \( \text{cod} : \text{Set} \rightarrow \downarrow \text{Set} \) with a fibration whose fibers are the homotopy categories of \( \text{Set} / X \).

It works! I.e., it is still a Heyting-fibration, with equality given by path spaces. (In fact, this works with \( \text{Top} \) as well!)
Homotopical semantics, fibrationally

The path space has this universal property “up to homotopy” (*).

\[
\begin{array}{ccc}
X & \xrightarrow{c} & X^I \\
X & \xrightarrow{\Delta X} & X \times X \\
E & \xrightarrow{} & \text{cod}
\end{array}
\]

Idea:
Homotopical semantics, fibrationally

The path space has this universal property “up to homotopy” (*).

\[
\begin{array}{ccc}
E & \xrightarrow{c} & X^I \\
\uparrow & & \downarrow \\
X & \xrightarrow{\Delta_X} & X \times X
\end{array}
\]

Idea: replace cod \[
\begin{array}{c}
sSet \\
\downarrow \\
sSet
\end{array}
\] with a fibration whose fibers are the homotopy categories of \(sSet/X\).
Homotopical semantics, fibrationally

The path space has this universal property “up to homotopy” (\(^*\)).

\[
\begin{align*}
X & \xrightarrow{c} X^I \\
X & \xrightarrow{\Delta^X} X \times X
\end{align*}
\]

Idea: replace \(\text{cod} \quad \downarrow\) with a fibration whose fibers are the homotopy categories of \(\text{sSet}/X\). It works!
Homotopical semantics, fibrationally

The path space has this universal property “up to homotopy” (*).

\[
\begin{array}{ccc}
\Delta^X & 
\rightarrow & X \\
\downarrow & & \downarrow \\
X & 
\rightarrow & X \times X
\end{array}
\]

Idea: replace \( \text{cod} \) with a fibration whose fibers are the \( \text{homotopy categories} \) of \( s\text{Set}/X \). It works! i.e., it is still a Heyting-fibration,
Homotopical semantics, fibrationally

The path space has this universal property “up to homotopy” \((*)\).

\[
\begin{align*}
E & \\
\uparrow & \\
X & \xrightarrow{c} X^I \\
\Delta X & \xrightarrow{X \times X}
\end{align*}
\]

Idea: replace cod \(\downarrow\) \(sSet\) with a fibration whose fibers are the homotopy categories of \(sSet/X\). It works! I.e., it is still a Heyting-fibration, with equality given by path spaces.
Homotopical semantics, fibrationally

The path space has this universal property “up to homotopy” (*).

\[ \begin{array}{ccc}
E & \rightarrow & X \\
\uparrow & & \downarrow c \\
I_X & \rightarrow & X^I \\
\Delta X & \rightarrow & X \times X \\
\end{array} \]

Idea: replace cod \( \text{sSet} \rightarrow \) with a fibration whose fibers are the homotopy categories of \( \text{sSet}/X \). It works! I.e., it is still a Heyting-fibration, with equality given by path spaces. (In fact, this works with \( \text{Top} \) as well!)
Homotopy invariance

How can we express homotopy invariance with this setup?

\[ M \xrightarrow{h} N \]

Partially answer:

\[ M(A) \rightarrow N(A) \]

\[ M(B) \rightarrow N(B) \]

\[ h(f) \simeq h(B) \]
Homotopy invariance

How can we express homotopy invariance with this setup?

(Partial) answer: $M \to N$ is a pseudo-natural transformation into the homotopy 2-category of simplicial sets.

$M(A) \to N(A)$

$M(B) \to N(B)$

$h_A \sim h_B$
Homotopy invariance

How can we express homotopy invariance with this setup?

\[
\begin{array}{ccc}
\text{Pf}_L & \xrightarrow{M} & \text{Ho(sSet)} \\
\downarrow & & \downarrow \\
\text{Ctx}_L & \xrightarrow{N} & \text{sSet}
\end{array}
\]
Homotopy invariance

How can we express homotopy invariance with this setup?

\[ \text{Pf}_\mathcal{L} \xrightarrow{\tilde{M}} \text{Ho}(\text{sSet} \rightarrow) \]

\[ \text{Ctx}_\mathcal{L} \xrightarrow{\tilde{N}} \text{sSet} \]

\[ \text{M}(A) \xrightarrow{\sim} \text{H}(B) \]

(Partial) answer: \( \tilde{M} \) is a pseudo-natural transformation into the homotopy 2-category of simplicial sets.

\[ \text{M}(A) \xrightarrow{\sim} \text{H}(B) \]
Homotopy invariance

How can we express homotopy invariance with this setup?

\[
\begin{array}{ccc}
\text{Pf}_\mathcal{L} & \xrightarrow{\tilde{M}} & \text{Ho}(sSet \rightarrow) \\
\downarrow & & \downarrow \\
\text{Ctx}_\mathcal{L} & \xrightarrow{\tilde{N}} & sSet
\end{array}
\]

(Partial) answer: \( \tilde{M} \) is a pseudo-natural transformation into the homotopy 2-category of simplicial sets.

\[ M(A) \xrightarrow{h_A} M(B) \sim \tilde{N}(A) \xrightarrow{h_B} \tilde{N}(B) \]
Homotopy invariance

How can we express homotopy invariance with this setup?

\[ \text{Pf}_\mathcal{L} \xrightarrow{\tilde{M}} \text{Ho}(\text{sSet}^{\rightarrow}) \]

\[ \text{Ctx}_\mathcal{L} \xrightarrow{\tilde{N}} \text{sSet} \]

(Partial) answer: \( \tilde{M} \) is a pseudo-natural transformation into the homotopy 2-category of simplicial sets.

\[ M(A) \cong N(A) \]

\[ M(B) \cong N(B) \]

\[ h_A \cong h_B \]
Homotopy invariance

How can we express homotopy invariance with this setup?

(Partial) answer:

\[
\begin{array}{ccc}
\text{Pf}_\mathcal{L} & \xrightarrow{\tilde{M}} & \text{Ho(sSet} \to) \\
\downarrow & & \downarrow \\
\text{Ctx}_\mathcal{L} & \xrightarrow{M} & \text{sSet}
\end{array}
\]
Homotopy invariance

How can we express homotopy invariance with this setup?

(Partial) answer: \( M \xrightarrow{?} N \) is a pseudo-natural transformation into the homotopy 2-category of simplicial sets.
Homotopy invariance

How can we express homotopy invariance with this setup?

(Partial) answer: \( M \rightarrow N \) is a \textit{pseudo-natural transformation} into the \textit{homotopy 2-category} of simplicial sets.

\[
\begin{align*}
M(A) & \xrightarrow{h_A} N(A) \\
M(f) & \Downarrow \phi \quad \Downarrow \\
M(B) & \xrightarrow{h_B} N(B)
\end{align*}
\]
1-discrete 2-fibrations
1-discrete 2-fibrations

Fibrations arise as pullbacks
1-discrete 2-fibrations

Fibrations arise as pullbacks of a universal fibration over $\text{Cat}^{\text{op}}$.
1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\text{op}}$.

\[
\begin{array}{ccc}
C & \xrightarrow{c} & \mathbf{B} \\
\downarrow & & \downarrow \\
\end{array}
\]
1-discrete 2-fibrations

Fibrations arise as pullbacks of a \textit{universal fibration} over $\text{Cat}^{\text{op}}$.

\[
\begin{array}{ccc}
C & \rightarrow & \text{Cat}^{\text{op}} \\
\downarrow C & \downarrow & \downarrow \\
\downarrow & B & \rightarrow \text{Cat}^{\text{op}}
\end{array}
\]
1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\text{Cat}^{\text{op}}$.

\[
\begin{array}{ccc}
\text{C} & \longrightarrow & \text{Cat}^{\text{op}} \\
\downarrow \sigma & & \downarrow \\
\text{B} & \longrightarrow & \text{Cat}^{\text{op}}
\end{array}
\]
1-discrete 2-fibrations

Fibrations arise as pullbacks of a universal fibration over $\text{Cat}^{\text{op}}$.

$$
\begin{array}{ccc}
\text{C} & \rightarrow & \text{Cat}^{\text{op}} \\
\downarrow \text{c} & & \downarrow \\
\text{B} & \rightarrow & \hat{\text{C}} \\
\end{array}
$$

Here, $\hat{\text{C}}$ is a pseudofunctor.
1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\text{Cat}^{\text{op}}$.

\[
\begin{array}{ccc}
\text{C} & \longrightarrow & \text{Cat}^{\text{op}} \\
\downarrow c & & \downarrow \\
\text{B} & \longrightarrow & \text{Cat}^{\text{op}} \\
\end{array}
\]

Here, $\hat{C}$ is a *pseudofunctor* and $\text{Cat}$ is considered as a 2-category.
1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\mathbf{Cat}^{\text{op}}$.

\[
\begin{array}{ccc}
\mathbf{C} & \longrightarrow & \mathbf{Cat}^{\text{op}} \\
\downarrow c & & \downarrow \\
\mathbf{B} & \xrightarrow{\hat{c}} & \mathbf{Cat}^{\text{op}}
\end{array}
\]

Here, $\hat{C}$ is a *pseudofunctor* and $\mathbf{Cat}$ is considered as a *2-category*. This still makes sense.
1-discrete 2-fibrations

Fibrations arise as pullbacks of a universal fibration over $\text{Cat}^{\text{op}}$.

\[
\begin{array}{ccc}
\text{C} & \longrightarrow & \text{Cat}^{\text{op}} \\
\downarrow c & & \downarrow \\
\text{B} & \xrightarrow{\widehat{c}} & \text{Cat}^{\text{op}}
\end{array}
\]

Here, $\widehat{C}$ is a pseudofunctor and $\text{Cat}$ is considered as a 2-category. This still makes sense when $\text{B}$ is also a 2-category.
1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\text{Cat}^{\text{op}}$.

\[
\begin{array}{ccc}
C & \longrightarrow & \text{Cat}^{\text{op}} \\
\downarrow^c & & \downarrow \\
B & \longrightarrow & \hat{\text{Cat}}^{\text{op}} \\
\end{array}
\]

Here, $\hat{\text{C}}$ is a *pseudofunctor* and $\text{Cat}$ is considered as a 2-category. This still makes sense when $B$ is also a 2-category.

The resulting notion
1-discrete 2-fibrations

Fibrations arise as pullbacks of a \textit{universal fibration} over $\textbf{Cat}^{\text{op}}$.

\[
\begin{array}{ccc}
\text{C} & \longrightarrow & \text{Cat}^{\text{op}} \\
\downarrow \quad \quad & \quad & \downarrow \\
\text{B} & \overset{\hat{c}}{\longrightarrow} & \text{Cat}^{\text{op}}
\end{array}
\]

Here, $\hat{C}$ is a \textit{pseudofunctor} and $\textbf{Cat}$ is considered as a 2-category. This still makes sense when $\text{B}$ is also a 2-category. The resulting notion is that of a \textit{1-discrete 2-fibration}.
1-discrete 2-fibrations

Fibrations arise as pullbacks of a \textit{universal fibration} over $\text{Cat}^{\text{op}}$.

\[
\begin{array}{c}
\text{C} \\
\downarrow c \\
\rightarrow \text{Cat}^{\text{op}} \\
\end{array}
\]

Here, $\widehat{\text{C}}$ is a \textit{pseudofunctor} and $\text{Cat}$ is considered as a 2-category. This still makes sense when $\text{B}$ is also a 2-category. The resulting notion is that of a \textit{1-discrete 2-fibration}, in which $\text{C}$ is (also) also a 2-category.
1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over \( \text{Cat}^{\text{op}} \).

\[
\begin{array}{ccc}
\text{C} & \longrightarrow & \text{Cat}^{\text{op}} \\
\downarrow^c & & \downarrow \\
\text{B} & \longrightarrow & \text{Cat}^{\text{op}} \\
\end{array}
\]

Here, \( \hat{\text{C}} \) is a *pseudofunctor* and \( \text{Cat} \) is considered as a *2-category*. This still makes sense when \( \text{B} \) is also a 2-category. The resulting notion is that of a *1-discrete 2-fibration*, in which \( \text{C} \) is (also) also a 2-category.

The pseudo-functor \( \text{sSet} \to \text{Cat}^{\text{op}} \)
1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over \( \text{Cat}^{\text{op}} \).

\[
\begin{array}{ccc}
\text{C} & \longrightarrow & \text{Cat}^{\text{op}} \\
\downarrow \circ & \downarrow \circ & \downarrow \circ \\
\text{B} & \longrightarrow & \text{Hat}^{\text{op}}
\end{array}
\]

Here, \( \hat{\text{C}} \) is a *pseudofunctor* and \( \text{Cat} \) is considered as a 2-category. This still makes sense when \( \text{B} \) is also a 2-category.

The resulting notion is that of a *1-discrete 2-fibration*, in which \( \text{C} \) is (also) also a 2-category.

The pseudo-functor \( \text{sSet} \to \text{Cat}^{\text{op}} \) associated to \( \text{Ho}(\text{sSet}^{\to}) \) extends to the 2-category \( \text{sSet} \), hence this fibration extends to a 1D2F.
1-discrete 2-fibrations

Fibrations arise as pullbacks of a *universal fibration* over $\text{Cat}^{\text{op}}$.

\[
\begin{array}{ccc}
\text{C} & \longrightarrow & \text{Cat}^{\text{op}} \\
\downarrow^{c} & & \downarrow \\
\text{B} & \longrightarrow & \hat{\text{C}} \\
& \uparrow^{\hat{c}} & \\
& & \text{Cat}^{\text{op}}
\end{array}
\]

Here, $\hat{\text{C}}$ is a *pseudofunctor* and $\text{Cat}$ is considered as a 2-category. This still makes sense when $\text{B}$ is also a 2-category. The resulting notion is that of a *1-discrete 2-fibration*, in which $\text{C}$ is (also) also a 2-category.

The pseudo-functor $\text{sSet} \to \text{Cat}^{\text{op}}$ associated to $\text{Ho}(\text{sSet}^{\rightarrow})$ extends to the 2-category $\text{sSet}$.
1-discrete 2-fibrations

Fibrations arise as pullbacks of a universal fibration over $\textbf{Cat}^{\text{op}}$.

\[
\begin{array}{ccc}
\text{C} & \longrightarrow & \text{Cat}^{\text{op}} \\
\downarrow_{c} & & \downarrow \\
\text{B} & \longrightarrow & \text{Cat}^{\text{op}}
\end{array}
\]

Here, $\hat{C}$ is a pseudofunctor and $\text{Cat}$ is considered as a 2-category. This still makes sense when $B$ is also a 2-category. The resulting notion is that of a 1-discrete 2-fibration, in which $C$ is (also) also a 2-category.

The pseudo-functor $\text{sSet} \rightarrow \text{Cat}^{\text{op}}$ associated to $\text{Ho}(\text{sSet} \rightarrow)$ extends to the 2-category $\text{sSet}$, hence this fibration extends to a 1D2F.
Homotopy invariance

The desired homotopy-invariance property then amounts to the existence of a pseudo-natural equivalence $\tilde{\alpha}$ over a given pseudo-natural equivalence $\alpha$.

This can again be shown from the freeness property of $\text{Pf} \downarrow \text{Ctx}$. 
Homotopy invariance

The desired homotopy-invariance property
Homotopy invariance

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Homotopy invariance

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\begin{equation}
\begin{array}{ccc}
Pf_{\mathcal{L}} & \xrightarrow{\tilde{\alpha}} & Ho(sSet) \\
\downarrow & \swarrow & \downarrow \\
Ctx_{\mathcal{L}} & \xrightarrow{\alpha} & sSet
\end{array}
\end{equation}
Homotopy invariance

The desired homotopy-invariance property then amounts to the existence of a pseudo-natural equivalence \( \tilde{\alpha} \) over a given pseudo-natural equivalence \( \alpha \).

\[
Pf_{\mathcal{L}} \xrightarrow{\tilde{\alpha} \simeq} \text{Ho}(sSet \to) \xrightarrow{\tilde{\alpha} \simeq} \text{sSet}
\]

This can again be shown from the freeness property of \( Pf_{\mathcal{L}} \xrightarrow{} \text{Ctx}_{\mathcal{L}} \).
Outline

1. A diagram
2. Propositions as spaces
3. Properties
4. Fibrational semantics
5. The abstract invariance theorem
The abstract invariance theorem

This argument depended heavily on the special nature of the category $sSet$. 

...
The abstract invariance theorem

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The abstract invariance theorem

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The abstract invariance theorem

This argument depended heavily on the special nature of the category \textbf{sSet}. (The \textit{isomorphism} invariance property, by contrast, does not.)
The abstract invariance theorem

This argument depended heavily on the special nature of the category $sSet$. (The *isomorphism* invariance property, by contrast, does not.) To put the proof in the proper, general context, we should

• Show that for any Heyting fibration $E \downarrow B$, there is natural 2-categorical structure on $B$
• (and that this recovers the usual one on $sSet$ and $Top$)
• Show that the associated pseudofunctor $B \to \text{Cat}^{\text{op}}$ is a pseudo-functor of 2-categories (thus giving us a 1D2F)
• We can do it!

In fact, we need much less than a Heyting fibration (a "$\land$-fibration" is good enough)

• The 2-categorical structure on $B$ is given by the "internal" notion of homotopy/equality
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The abstract invariance theorem

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- We can do it! In fact, we need much less than a Heyting fibration (a \textit{\&=}\text{-fibration}” is good enough)
- The 2-categorical structure on $B$ is given by the “internal” notion of homotopy/equality
Thank you for your attention!

For more information, see:

- Homotopies in Grothendieck fibrations (arXiv:1905.10690)
- First-order homotopical logic (forthcoming)