Contextual categories as monoids in a category of collections

(Work in progress)

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HoTT 2019, Pittsburgh

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We want to give a good, algebraic description of a **theory** expressed in the **language** of Martin-Löf’s framework of dependent types.

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Problem

A theory is a *syntactic* object, and these don’t obviously have a nice algebraic definition.

Well-known syntactic definitions of what such a theory should be are GATs [Car78] and FOLDS signatures [Pal16].

Disclaimer

We don't consider any *type formers* (Id, Π, U, etc.) in our theories — i.e. the syntactic category of a "dependently typed theory" will simply be a contextual category with no additional structure.

(Eventually, we'd like to add them one by one.)

We'd like:

- A good category with a nice description (not explicitly involving any syntax).
- But each of whose objects corresponds canonically to a syntactic dependently typed theory (and the same for morphisms).

A motivating example is the category of symmetric $\mathcal{S}\mathrm{et}$-operads, which correspond to certain algebraic theories.

Our proposal for a category of theories

Recall

A contextual category is a small category C "resembling" the syntactic category of a dependently typed theory.

Our proposal for a category of theories

Our proposed definition

A theory is an I-**contextual category**, where I is a finitely branching inverse category (I is the *type signature* of the theory).

The category $\mathcal{C}\mathrm{xlCat}(I)$ of these embeds into the category of contextual categories under the *free contextual category on I*.

$$\mathcal{C}\mathrm{xlCat}(I) \hookrightarrow C(I)/\mathcal{C}\mathrm{xlCat}$$

Nice features

- $\mathcal{C}\mathrm{xlCat}(I)$ is the category of monoids in a presheaf category of "I-coloured collections" (analogous to operads and polynomial monads).
- From any $T \in \mathcal{C}\mathrm{xlCat}(I)$, we can recover a syntax that presents it (its underlying collection).

Drawback

May not encompass all generalised algebraic theories.

Goals of this talk

1. Justify the following:

A **dependently typed theory** or I-**contextual category** is the data of

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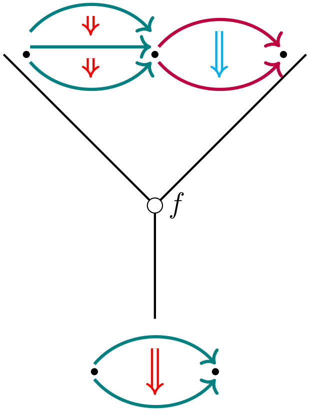
Example/particular case

A **multisorted Lawvere theory** is the data of

1. a set S (always a fin. branching inverse category)
2. and a finitary monad on $\mathcal{S}\mathrm{et}^S$.

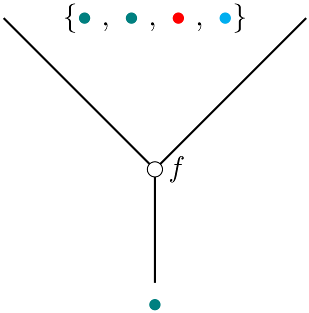
2. To convey the picture:



Every operation in a dependently typed theory takes a finite cell complex as input, and outputs a cell.

(This is related to Burroni-Leinster T-operads.)

Example/particular case



An operation in a multisorted Lawvere theory takes a finite coproduct of points as input, and outputs a point.

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- $$\begin{tikzcd} G_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ G_0 \end{tikzcd} \qquad \mathbb{G}^{\mathrm{op}} = \begin{tikzcd} \vdots \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ G_2 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ G_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ G_0 \end{tikzcd} \qquad \mathbb{O}^{\mathrm{op}} \text{ (opetopes).}$$

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- The free-*weak*-ω-category monad on $\mathcal{S}\mathrm{et}^{\mathbb{G}^{\mathrm{op}}}$.
- For $T : \mathcal{S}\mathrm{et}^I \to \mathcal{S}\mathrm{et}^I$ a finitary cartesian monad, every T-operad (à la Burroni-Leinster).
- And many more...

Syntactic example

Let $I = \{G_2 \rightrightarrows G_1 \rightrightarrows G_0\}$ with the (co)globular relations. Then I corresponds to the following type signature.

$$\vdash G_0 \qquad x, y : G_0 \vdash G_1(x, y) \qquad x, y : G_0, f, g : G_1(x, y) \vdash G_2(f, g)$$

The theory of 2-categories (or even of bicategories) is a collection of terms and definitional equalities expressible in this type signature.

Preliminaries

- Let I be a small category.

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- Recall that $\mathrm{Fin}(I)$ is the finite-colimit completion of I^{op}. When I is a set, $\mathrm{Fin}(I)$ is the also the finite-coproduct completion of I.

Cartesian collections

The presheaf category

$$\mathcal{C}\mathrm{oll}_I := \mathcal{S}\mathrm{et}^{I \times \mathrm{Fin}(I)}$$

is called the category of I-**collections**.

(Intuition: $F \in \mathcal{C}\mathrm{oll}_I$ should be thought of as a *term signature* — for each *context* $\Gamma \in \mathrm{Fin}(I)$ and each *sort* $i \in I$, $F(i, \Gamma)$ is the set of operations with input Γ and output sort i.)

Composition of cartesian collections

I-collections can be composed via **substitution**:

$$G \circ F(i, \Gamma) := \int^{\Theta \in \mathrm{Fin}(I)} G(i, \Theta) \times \mathcal{S}\mathrm{et}^I(\Theta, F(-, \Gamma)).$$

$(\mathcal{C}\mathrm{oll}_I, \circ, E)$ is a (non-symmetric) **monoidal category**, where $E : \mathrm{Fin}(I) \hookrightarrow \mathcal{S}\mathrm{et}^I$.

Cartesian collections and endofunctors on $\mathcal{S}\mathrm{et}^I$

The functor $\mathrm{Lan}_E(-) : \mathcal{C}\mathrm{oll}_I \to [\mathcal{S}\mathrm{et}^I, \mathcal{S}\mathrm{et}^I]$ of left Kan extension along $E : \mathrm{Fin}(I) \hookrightarrow \mathcal{S}\mathrm{et}^I$ is (1) **fully faithful** and (2) **monoidal**.

$$\begin{tikzcd} \mathrm{Fin}(I) \arrow[r, "F"] \arrow[d, hook, "E"'] \arrow[dr, phantom, "\cong"] & \mathcal{S}\mathrm{et}^I \\ \mathcal{S}\mathrm{et}^I \arrow[ur, "\mathrm{Lan}_E\, F"'] & \end{tikzcd} \tag{1}$$

$$\mathrm{Lan}_E(F \circ G) \;\cong\; \mathrm{Lan}_E\, F \circ \mathrm{Lan}_E\, G \quad ; \quad \mathrm{Lan}_E\, E \;\cong\; \mathrm{id} \tag{2}$$

Consequence

$$\mathrm{Lan}_E - : \mathcal{M}\mathrm{on}(\mathcal{C}\mathrm{oll}_I, \circ, E) \hookrightarrow \mathcal{M}\mathrm{nd}(\mathcal{S}\mathrm{et}^I)$$

The category of monoids in $\mathcal{C}\mathrm{oll}_I$ is a full subcategory of the category of monads on $\mathcal{S}\mathrm{et}^I$. It is none other than the category of **finitary monads** on $\mathcal{S}\mathrm{et}^I$.

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Remarks

- We have only used that I is a small category.
- $\mathcal{M}\mathrm{on}(\mathcal{C}\mathrm{oll}_I, \circ, E)$is also known as the category of *monads with arities* (Weber) or *Lawvere theories with arities* (Melliès) for the arities $E : \mathrm{Fin}(I) \hookrightarrow \mathcal{S}\mathrm{et}^I$.

Contextual categories as monoids in collections

Inverse categories

Definition

An **inverse category** is:

- a small category I,
- whose objects are graded by "dimension" $dim : \mathrm{Ob}(I) \to \mathcal{O}\mathrm{rd}$,
- such that non-identity morphisms strictly decrease dimension,
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I is **finitely branching** if the tree $i/_I$ generated by every $i \in I$ is finite.

Main observation

Proposition (L.S., LeFanu Lumsdaine)

Let I be a finitely branching inverse category. Then $\mathrm{Fin}(I)^{\mathrm{op}}$ is equivalent to a contextual category $C(I)$ (the free contextual category on I).

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(Note: The structure of a contextual category *does not* transfer across an equivalence of categories.)

- Particular case: I is a set, then $\mathrm{Fin}(I)^{\mathrm{op}}$ is the free finite-product category on I.

Proof:

1. Note that:

   - The Yoneda embedding factors as $\mathbf{y} : I^{\mathrm{op}} \hookrightarrow \mathrm{Fin}(I) \hookrightarrow \mathcal{S}\mathrm{et}^{I}$.

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- The boundary inclusions $\partial i \hookrightarrow \mathbf{y}i$ are finitely presentable (since i/I is finite).
- Every finite cell complex in $\mathcal{S}\mathrm{et}^I$ is finitely presentable.
- Every $X \in \mathrm{Fin}(I)$ can be written as a finite cell complex.

2. Define a **cell context** to be a finite sequence

$$\emptyset \to X_1 \to X_2 \to \ldots \to X$$

of *chosen* pushouts of boundary inclusions:

$$\begin{tikzcd} \partial i \arrow[r] \arrow[d, hook] & X_n \arrow[d, hook] \\ \mathbf{y}i \arrow[r] & X_{n+1}. \end{tikzcd}$$

Definition

The category Cell_I has as objects the cell contexts and as morphisms, $\mathrm{Cell}_I(\emptyset \ldots \to X, \emptyset \ldots \to Y) := \mathcal{S}\mathrm{et}^I(X, Y)$.

Clearly, $\mathrm{Cell}_I \simeq \mathrm{Fin}(I)$.

3. Not hard to see that $C(I) := \mathrm{Cell}_I^{\mathrm{op}}$ is a contextual category. (In fact, it is the *free contextual category on* I.) $\square$

Remarks

- A collection $X \in \mathcal{C}\mathrm{oll}_I \simeq \mathcal{S}\mathrm{et}^{I \times \mathrm{Cell}_I}$ is now *literally* an I-sorted term signature.
- $C(I)$ has all finite limits.

I-contextual categories

Definition

An I-**contextual category** is a morphism of contextual categories $F : C(I) \to D$ such that in the (identity-on-objects, fully faithful) factorisation

$$\begin{tikzcd} C(I) \arrow[r, "F_1"] \arrow[rr, bend right, "F"'] & D_I \arrow[r, hook, "F_2"] & D \end{tikzcd}$$

$F_2 : D_I \hookrightarrow D$ exhibits D as the *contextual completion* of D_I.

A morphism of I-contextual categories is a morphism in the coslice $C(I)/\mathcal{C}\mathrm{xlCat}$.

Theorem (L.S., LeFanu Lumsdaine)

The following categories are equivalent:

1. *The category* $\mathcal{C}\mathrm{xl}\mathrm{Cat}(I)$ *of* I*-contextual categories.*
2. *The category* $\mathcal{M}\mathrm{on}(\mathcal{C}\mathrm{oll}_I, \circ, E)$ *of monoids in* I*-sorted cartesian collections.*
3. *The category of finitary monads on* $\mathcal{S}\mathrm{et}^I$*.*

Proof.

Make use of the theory of Lawvere theories with arities [Mel10], [BMW12]. $\square$

Summary, current and future work

- We introduce I-**contextual categories** as algebraic objects (monoids in collections) with an underlying dependently typed theory.
- We are working on a *linear* variant of this, and hoping to get a definition of *dependently coloured symmetric operad*/*linear dependently typed theory*.
- The "base change" properties of I-contextual categories remain to be understood.
- We would eventually like to add Id-types to this formalism.

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