Good Fibrations through the Modal Prism

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August 15, 2019
Homotopy theory is the study of the ways things can be identified:

“The algebra of the ambiguity in how things are identified.”
Plan of the Talk

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Algebraic Topology is the study of the connectivity of space:

“We may identify points by giving continuous paths between them.”
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- Book HoTT is a great language to do homotopy theory, but there is no way to say that one type is the homotopy type of another type:
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In Book HoTT, we can do homotopy theory, but not algebraic topology.
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- Book HoTT is a great language to do homotopy theory, but there is no way to say that one type is the homotopy type of another type:

In Book HoTT, we can do homotopy theory, but not algebraic topology.

- To fix this, Shulman adds a system of (co)modalities including the shape modality $\int$ which sends a type to its homotopy type. (Real Cohesive HoTT)
Plan of the Talk

In this talk, we’ll see a modal notion of \textit{fibration}, suitable for synthetic algebraic topology.
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- In this talk, we’ll see a modal notion of *fibration*, suitable for synthetic algebraic topology.
- We find this notion of modal fibration by looking at functions through the *modal prism*.
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- In this talk, we’ll see a modal notion of *fibration*, suitable for synthetic algebraic topology.
- We find this notion of modal fibration by looking at functions through the *modal prism*.
- Finally, we’ll see a trick for showing that maps are $ʃ$-fibrations.
- We’ll use this trick to calculate the fundamental group of the circle without using higher inductive types, and classify the $n$-fold covers of the circle.
A modality is a way of changing what it means to identify two elements.

- A type $X$ is $\triangleright$-modal if $(-)^\triangleright : X \to \triangleright X$ is an equivalence.
A *modality* is a way of changing what it means to identify two elements.

- A type $X$ is *!-modal* if $(-)^! : X \rightarrow !X$ is an equivalence.
- When mapping out of $!X$ into a modal type $Z$, it suffices to map out of $X$.

\[
\begin{array}{c}
X \xrightarrow{(-)^!} !X \\
\downarrow g \\
\downarrow \text{ind} \downarrow g \\
\downarrow \quad \downarrow \quad \downarrow \\
Z
\end{array}
\]
A **modality** is a way of changing what it means to identify two elements.

- A type $X$ is **$!$-modal** if $(-)!: X \to \,!X$ is an equivalence.
- When mapping out of $!X$ into a modal type $Z$, it suffices to map out of $X$.

$$
\begin{array}{ccc}
X & \xrightarrow{(\cdot)!} & !X \\
\downarrow g & & \downarrow \text{ind}!g \\
\downarrow \text{ind}!g & & \\
Z & & \\
\end{array}
$$

- In particular, for any function $f : X \to Y$ we get a function $!f : !X \to !Y$ and a naturality square:

$$
\begin{array}{ccc}
X & \xrightarrow{(\cdot)!} & !X \\
\downarrow f & & \downarrow !f \\
Y & \xrightarrow{(\cdot)!} & !Y \\
\end{array}
$$
The Modal Prism

\[ \text{fib}_f(y) \xrightarrow{\delta} \text{fib}_f(y^!) \]

\[ \downarrow \quad \downarrow \]

\[ X \xrightarrow{(-)^!} !X \]

\[ \downarrow \quad \downarrow \]

\[ Y \xrightarrow{(-)^!} !!Y \]
The Modal Prism

\[ \text{fib}_f(y) \xrightarrow{(-)'} \text{fib}!_f(y') \]

\[ \text{X} \xrightarrow{(-)'} \text{!X} \]

\[ \text{Y} \xrightarrow{(-)'} \text{!Y} \]
The Modal Prism

The map \( f : X \rightarrow Y \) is \(!\)-modal if \((-)!\) is an equivalence, \(!\)-connected if \(! \text{fib}_f(y)\) is contractible, \(!\)-etale if \(\delta\) is an equivalence, \(!\)-equivalence if \(! \text{fib}_f(y_!')\) is contractible, \(\text{UFP}, \text{RSS}\) if \(a !\)-equivalence if \(! \text{fib}_f(y_!')\) is contractible, \(S_\infty, W, R, RW\) if \(\gamma\) is an equivalence for all \(y : Y\).
The map $f : X \to Y$ is

- \textit{!-modal} if $(-)^!$ is an equivalence
- \textit{!-connected} if $!\text{fib}_f(y)$ is contractible

\[ \begin{array}{ccc}
\text{fib}_f(y) & \xrightarrow{\delta} & \text{fib}!_f(y^!)
\\
\downarrow & & \downarrow
\\
(-)^! & & !\text{fib}_f(y)
\end{array} \]

\{ UFP, RSS \}
The map $f : X \to Y$ is

- \textit{!-modal} if $(-)!$ is an equivalence
- \textit{!-connected} if $!\fib_f(y)$ is contractible
- \textit{!-étale} if $\delta$ is an equivalence
- a \textit{!-equivalence} if $\fib!_f(y!)$ is contractible

\[
\begin{array}{ccc}
\fib_f(y) & \overset{\delta}{\longrightarrow} & \fib!_f(y!)
\end{array}
\]

\[
\begin{array}{cc}
(-)! & \to & !\fib_f(y)
\end{array}
\]
The map $f : X \to Y$ is

- !-
  - modal if $(-)!$ is an equivalence
  - connected if $! \text{fib}_f(y)$ is contractible

- !-étale if $\delta$ is an equivalence

- !-equivalence if $\text{fib}_{!f}(y^!)$ is contractible

- a !-fibration if $\gamma$ is an equivalence

for all $y : Y$. 
The Two Factorization Systems

\[ X \times \text{fib} \left( y \times \text{fib} \right) \]
The Two Factorization Systems

\[(y : Y) \times \text{fib}_f(y)\]

\[
\downarrow \text{fst}
\]

\[Y\]
The Two Factorization Systems

\[(y : Y) \times \text{fib}_f(y)\]

\[
\begin{align*}
\text{tot}(\vdash) & \quad \text{tot}(\delta) \\
(y : Y) \times !\text{fib}_f(y) & \quad (y : Y) \times \text{fib}_f(y) \\
Y & \\
\end{align*}
\]
The Two Factorization Systems

\[(y : Y) \times \text{fib}_f(y)\]

!-connected

\[(y : Y) \times \text{fib}_f(y)\]

\[(y : Y) \times \text{fib}_f(y)\]

!-modal

\[(y : Y) \times \text{fib}_f(y)\]

\[(y : Y) \times \text{fib}_f(y)\]

!-étale

\[(y : Y) \times \text{fib}_f(y)\]
The Two Factorization Systems

\[(y : Y) \times \text{fib}_f(y)\]

\[-\text{connected}\]

\[-\text{equivalence}\]

\[\text{tot}(\gamma)\]

\[-\text{modal}\]

\[-\text{étale}\]

\[Y\]

\[(y : Y) \times \text{fib}_f(y)\]

\[(y : Y) \times \text{fib}_f(y)\]
Modal Fibrations

If

\[ \text{fib}_f \rightarrow E \xrightarrow{f} B \]

is a fiber sequence, then \( \gamma \) is the comparison map

\[ \begin{array}{c}
\text{fib}_f \\
\downarrow \gamma \\
\text{fib}_! f
\end{array} \xrightarrow{f} \begin{array}{c}
!E \\
\downarrow \! f \\
!B
\end{array} \]

An \( S \)-fibration resembles the classical Dold-Thom notion of quasi-fibration.
Modal Fibrations

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\downarrow \\
\text{fib}_! f
\end{array} \quad \begin{array}{c}
\gamma \\
\downarrow
\end{array} \quad \begin{array}{c}
! E \\
\downarrow
\end{array} \quad \begin{array}{c}
! E \\
\downarrow
\end{array} \quad \begin{array}{c}
! f \\
\downarrow
\end{array} \quad \begin{array}{c}
! B
\end{array} \]

A map \( f : E \rightarrow B \) is a \(!\)-fibration if and only if \( ! \) preserves all its fibers.

An \( \int \)-fibration resembles the classical Dold-Thom notion of quasi-fibration.
The Fundamental Group of the Circle

If we knew that the map \((\cos, \sin) : \mathbb{R} \to S^1\) were a \(\int\)-fibration, then the fiber sequence

\[ \mathbb{Z} \to \mathbb{R} \to S^1 \]

would give us a fiber sequence on homotopy types:

\[ \int \mathbb{Z} \to \int \mathbb{R} \to \int S^1. \]
The Fundamental Group of the Circle

If we knew that the map \((\cos, \sin) : \mathbb{R} \to S^1\) were a \(\mathbb{S}\)-fibration, then the fiber sequence

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\mathbb{Z} \to \mathbb{R} \to S^1
\]

would give us a fiber sequence on homotopy types:

\[
\mathbb{Z} \to \ast \to \int S^1.
\]

This calculates the loop space of the circle without using higher inductive types.
Properties of Modal Fibrations

Theorem

For a map \( f : X \to Y \), the following are equivalent:

1. \( f \) is a \( ! \)-fibration,
Properties of Modal Fibrations

Theorem

For a map $f : X \to Y$, the following are equivalent:

1. $f$ is a $!$-fibration,
2. The two factorizations of $f$ agree,
3. The $!$-modal factor of $f$ is $!$-étale,
4. The $!$-equivalence factor of $f$ is $!$-connected,
**Theorem**

For a map $f : X \rightarrow Y$, the following are equivalent:

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2. The two factorizations of $f$ agree,
3. The $!$-modal factor of $f$ is $!$-étale,
4. The $!$-equivalence factor of $f$ is $!$-connected,
5. $!$ preserves all pullbacks along $f$,
Properties of Modal Fibrations

THEOREM

For a map \( f : X \to Y \), the following are equivalent:

1. \( f \) is a \(!\)-fibration,
2. The two factorizations of \( f \) agree,
3. The \(!\)-modal factor of \( f \) is \(!\)-étale,
4. The \(!\)-equivalence factor of \( f \) is \(!\)-connected,
5. \! preserves all pullbacks along \( f \),
6. \( f \) has “\(!\)-locally constant \(!\)-fibers”.

\[
\begin{align*}
(y : Y) \times \text{fib}_f(y) & \cong (y : Y) \times \text{fib}_f(y) \\
\text{!-equivalence} & \quad \text{tot}(\gamma) \quad \text{!-connected}
\end{align*}
\]
<table>
<thead>
<tr>
<th><strong>Corollary</strong></th>
</tr>
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<tbody>
<tr>
<td>!-fibrations are closed under composition and pullback.</td>
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</table>
## Corollary

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The pullback of a !-equivalence along a !-fibration is a !-equivalence.
A Modality is Lex along its Fibrations

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For a modality !, the following are equivalent:

1. ! is lex – it preserves all pullbacks,
A Modality is Lex along its Fibrations

**Corollary**

!-fibrations are closed under composition and pullback.

**Corollary**

The pullback of a !-equivalence along a !-fibration is a !-equivalence.

**Corollary**

For a modality !, the following are equivalent:

1. ! is lex – it preserves all pullbacks,
2. Every map is a !-fibration
3. The object classifier \( \text{Type}_* \to \text{Type} \) is a !-fibration.
A Modality is Lex along its Fibrations

Corollary

!-fibrations are closed under composition and pullback.

Corollary

The pullback of a !-equivalence along a !-fibration is a !-equivalence.

Corollary

For a modality !, the following are equivalent:

1. ! is lex – it preserves all pullbacks,
2. Every map is a !-fibration
3. The object classifier $\text{Type}_* \to \text{Type}$ is a !-fibration.
4. If each map in a family is a !-fibration, then the total map is a !-fibration,
5. For any map, the connecting map $\text{tot}(\gamma)$ between its factorizations is a !-fibration.
Showing Maps are Modal Fibrations

How do we know that \((\cos, \sin) : \mathbb{R} \to S^1\) is a \(\mathcal{f}\)-fibration?
Showing Maps are Modal Fibrations

How do we know that \((\cos, \sin) : \mathbb{R} \to S^1\) is a \(j\)-fibration?

A map \(f\) is a \(!\)-fibration if and only if it has “\(!\)-locally constant \(!\)-fibers”.
How do we know that \((\cos, \sin) : \mathbb{R} \to S^1\) is a \(\_\)-fibration?

A map \(f\) is a \(!\)-fibration if and only if it has “\(!\)-locally constant \(!\)-fibers”.

**Theorem**

A map \(f : X \to Y\) is a \(!\)-fibration if and only if \(\text{fib}_f\) factors through \(!Y\):

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{fib}_f} & \text{Type} \\
\downarrow (-)! & & \downarrow ! \\
\downarrow \downarrow & & \downarrow \\
\text{!}Y & \to & \text{Type}_! \\
\end{array}
\]
How do we know that \((\cos, \sin) : \mathbb{R} \to S^1\) is a \(\neg\)-fibration?

A map \(f\) is a \(!\)-fibration if and only if it has “\(!\)-locally constant \(!\)-fibers”.

**Theorem**

A map \(f : X \to Y\) is a \(!\)-fibration if and only if \(\text{fib}_f\) factors through \(!\ Y\):

\[
\begin{array}{c}
\text{Type} \\
\downarrow \text{!} \\
\text{Type}_! \\
\end{array}
\]

If \(f\) is a \(!\)-fibration, then we take \(\text{fib}_{!f} : ! Y \to \text{Type}_!\) as the factorization.
Showing Maps are \( \mathcal{f} \)-Fibrations

The shape modality \( \mathcal{f} \) has a right adjoint comodality \( \flat \), so we can use a trick.

**Lemma**

If \( X :: \text{Type} \) is *locally discrete* (\( \mathcal{f} \)-separated), then for \( x :: X \),

\[
\text{BAut}_X(x) :: (y : X) \times ||x = y||
\]

is discrete.

**Corollary**

If \( G :: \text{Type} \) is a discrete (\( \infty \)-)group, then \( \text{B}G \) is also discrete.

**Corollary**

If \( G :: \text{Type} \) is an (\( \infty \)-)group, then \( \text{B}S_G = S\text{B}G \).
Showing Maps are $\int$-Fibrations

The shape modality $\int$ has a right adjoint comodality $♭$, so we can use a trick.

Lemma

If $X :: \text{Type}$ is locally discrete ($\int$-separated), then for $x :: X$,

$$\text{BAut}_X(x) \equiv (y : X) \times \|x = y\|$$

is discrete.

Corollary

If $G :: \text{Type}$ is a discrete ($\infty$-)group, then $B G$ is also discrete.
Showing Maps are $\int$-Fibrations

The shape modality $\int$ has a right adjoint comodality $♭$, so we can use a trick.

**Lemma**

If $X :: Type$ is locally discrete ($\int$-separated), then for $x :: X$,

$$\text{BAut}_X(x) :\equiv (y : X) \times \| x = y \|$$

is discrete.

**Corollary**

If $G :: Type$ is a discrete ($\infty$-)group, then $BG$ is also discrete.

**Corollary**

If $G :: Type$ is an ($\infty$-)group, then $B \int G = \int BG$. 
Characterizing ∫-Fibrations

As a corollary, functions whose fibers have merely constant homotopy type are fibrations.

**Theorem**

Let \( f : E \to B \). If there is a \( F :: Type_f \) such that for all \( b : B \), we have \( \| F = \int \text{fib}_f(b) \| \), then \( f \) is a \( ∫ \)-fibration.
Characterizing \( \int \)-Fibrations

As a corollary, functions whose fibers have merely constant homotopy type are fibrations.

**Theorem**

Let \( f : E \to B \). If there is a \( F :: \text{Type}_\int \) such that for all \( b : B \), we have \( \| F = \int \text{fib}_f(b) \| \), then \( f \) is a \( \int \)-fibration.

**Proof.**

- Since \( F \) is a crisp element of a locally discrete type, \( \text{BAut}(F) \) is discrete.
Characterizing $\int$-Fibrations

As a corollary, functions whose fibers have merely constant homotopy type are fibrations.

**Theorem**

Let $f : E \to B$. If there is a $F :: \text{Type}_\int$ such that for all $b : B$, we have $\|F = \int \text{fib}_f(b)\|$, then $f$ is an $\int$-fibration.

**Proof.**

- Since $F$ is a crisp element of a locally discrete type, $BAut(F)$ is discrete.
- By hypothesis, $\int \text{fib}_f : B \to \text{Type}_\int$ factors through $BAut(F)$ and so also through $(-)\int : B \to \int B$. 
Characterizing \( \int \)-Fibrations

As a corollary, functions whose fibers have merely constant homotopy type are fibrations.

**Theorem**

Let \( f : E \to B \). If there is a \( F :: \text{Type}_\int \) such that for all \( b : B \), we have \( \| F = \int \text{fib}_f(b) \| \), then \( f \) is a \( \int \)-fibration.

**Proof.**

- Since \( F \) is a crisp element of a locally discrete type, \( \text{BAut}(F) \) is discrete.
- By hypothesis, \( \int \text{fib}_f : B \to \text{Type}_\int \) factors through \( \text{BAut}(F) \) and so also through \( (-)\int : B \to \int B \).
- So, \( \int \text{fib}_f \) is locally constant, and therefore \( f \) is a \( \int \)-fibration.
Examples of $\int$-Fibrations

**Theorem**

Let $f : E \to B$. If there is a $F :: \text{Type}_\int$ such that for all $b : B$, we have $\|F = \int \text{fib}_f(b)\|$, then $f$ is a $\int$-fibration.

**Motto**

If you were comfortable writing

“$\text{F} \to \text{E} \xrightarrow{f} \text{B}$”,

or talking about “the fiber $F$”, then $f$ is a fibration.
Examples of $\int$-Fibrations

Theorem

Let $f : E \to B$. If there is a $F :: \text{Type}_\int$ such that for all $b : B$, we have $\|F = \int \text{fib}_f(b)\|$, then $f$ is a $\int$-fibration.

Motto

If you were comfortable writing

"$F \to E \overset{f}{\to} B$",

or talking about "the fiber $F$", then $f$ is a fibration.

- $(\cos, \sin) : \mathbb{R} \to S^1$, with $F :\equiv \mathbb{Z}$,
- The Hopf fibration $h : S^3 \to S^2$, with $F :\equiv \int S^1$, and other Hopf-style fibrations,
- The Serre fibration $s : \text{SO}(3) \to S^2$, with $F :\equiv \int \text{SO}(2)$
Definition (Wellen)

A covering is a $\int_1$-étale map $c : E \to B$ whose fibers are sets, where $\int_1$ is the modality whose modal types are discrete groupoids.
## Classifying the Covers of the Circle, Modally

### Definition (Wellen)

A *covering* is a $\int_1$-étale map $c : E \to B$ whose fibers are sets, where $\int_1$ is the modality whose modal types are discrete groupoids.

### Corollary

Let $c : E \to B$. If there is a $F :: \text{Set}_\int$ such that for all $b : B$, we have $\|F = \text{fib}_f(b)\|$, then $c$ is a covering.
**Classifying the Covers of the Circle, Modally**

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A *covering* is a $\int_1$-étale map $c : E \to B$ whose fibers are sets, where $\int_1$ is the modality whose modal types are discrete groupoids.

**Corollary**

Let $c : E \to B$. If there is a $F :: \textbf{Set}_\int$ such that for all $b : B$, we have $\|F = \text{fib}_f(b)\|$, then $c$ is a covering.

**Definition**

An *$n$-fold covering* $c : E \to B$ is a map whose fibers have $n$ elements.
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A *covering* is a $\int_1$-étale map $c : E \to B$ whose fibers are sets, where $\int_1$ is the modality whose modal types are discrete groupoids.

**Corollary**

Let $c : E \to B$. If there is a $F :: \text{Set}_\int$ such that for all $b : B$, we have $\|F = \text{fib}_f(b)\|$, then $c$ is a covering.

**Definition**

An *$n$-fold covering* $c : E \to B$ is a map whose fibers have $n$ elements.

**Question**

What are the $n$-fold covers of the circle $S^1$?
An $n$-fold cover with an identification of a fiber with \{1, \ldots, n\} is a pointed map $C : S^1 \to \text{BAut}(n)$. 
Classifying the Covers of the Circle, Modally

- An $n$-fold cover with an identification of a fiber with $\{1, \ldots, n\}$ is a pointed map $C : S^1 \rightarrow BAut(n)$.
- Since $\{1, \ldots, n\}$ is discrete, so is $BAut(n)$ and therefore $C$ factors uniquely through $\int S^1$. 

\[
\begin{array}{ccc}
S^1 & \xrightarrow{C} & BAut(n) \\
\downarrow (-) \int & & \\
\int S^1 & \xrightarrow{\phantom{C}} & \phantom{BAut(n)}
\end{array}
\]
Classifying the Covers of the Circle, Modally

- An \( n \)-fold cover with an identification of a fiber with \( \{1, \ldots, n\} \) is a pointed map \( C : S^1 \to B\text{Aut}(n) \).
- Since \( \{1, \ldots, n\} \) is discrete, so is \( B\text{Aut}(n) \) and therefore \( C \) factors uniquely through \( \int S^1 \).

\[
\begin{array}{ccc}
S^1 & \xrightarrow{C} & B\text{Aut}(n) \\
\downarrow \phi & & \\
\int S^1 & \xrightarrow{(-)\phi} & B\text{Aut}(n)
\end{array}
\]

- But \( \int S^1 \) is a \( B\mathbb{Z} \), so this corresponds to a homomorphism \( \phi : \mathbb{Z} \to \text{Aut}(n) \): a permutation of \( n \) elements.
Classifying the Covers of the Circle, Modally

- It looks as though the connected components of the total space correspond to the cycle type of the permutation. Can we prove this?
Classifying the Covers of the Circle, Modally

- It looks as though the connected components of the total space correspond to the cycle type of the permutation. Can we prove this?

\[ \mathbb{S}^1 \xrightarrow{\mathcal{C}} \text{BAut}(n) \]

\[ \int \mathbb{S}^1 \xrightarrow{(-)\mathcal{C}} \text{B}_{\varphi} \]
Classifying the Covers of the Circle, Modally

• It looks as though the connected components of the total space correspond to the cycle type of the permutation. Can we prove this?

\[ S^1 \xrightarrow{C} \text{BAut}(n) \xrightarrow{(-)^f} B\varphi \]

• The cycle type is the set of orbits of the action of \( \varphi \) on the fiber, or

\[ \| (t : \int S^1) \times B\varphi(t) \|_0. \]
Classifying the Covers of the Circle, Modally

\[
S^1 \xrightarrow{c} B\text{Aut}(n)
\]
\[
(−) \downarrow \xrightarrow{B\varphi} \int S^1
\]

The square is a pullback and the bottom map $S^1$-connected, so the top map is as well. Therefore, we get an equivalence $S^1 \times C(s) \simeq (u : S^1 \times B\varphi(u))$ and so an equivalence on their 0-truncations.
Classifying the Covers of the Circle, Modally

\[(s : \mathbb{S}^1) \times C(s) \longrightarrow (u : \int \mathbb{S}^1) \times B\varphi(u)\]

\[
\begin{array}{ccc}
\mathbb{S}^1 & \longrightarrow & \int \mathbb{S}^1 \\
\downarrow & & \downarrow \\
\mathbb{S}^1 & \longrightarrow & \int \mathbb{S}^1 \\
& \longrightarrow & \\
\end{array}
\]

The square is a pullback and the bottom map is connected, so the top map is as well. Therefore, we get an equivalence and so an equivalence on their 0-truncations.
Classifying the Covers of the Circle, Modally

\[(s : \mathbb{S}^1) \times C(s) \rightarrow (u : \int \mathbb{S}^1) \times B\varphi(u)\]

The square is a pullback and the bottom map \(\int\)-connected, so the top map is as well.
Classifying the Covers of the Circle, Modally

\[(s : \mathbb{S}^1) \times C(s) \longrightarrow (u : \int \mathbb{S}^1) \times B\varphi(u)\]

\[
\begin{array}{ccc}
\mathbb{S}^1 & \longrightarrow & \int \mathbb{S}^1 \\
\downarrow & & \downarrow \ (\cdot)^{\text{f}} \\
\mathbb{S}^1 & \longrightarrow & \int \mathbb{S}^1
\end{array}
\]

The square is a pullback and the bottom map \(\int\)-connected, so the top map is as well. Therefore, we get an equivalence

\[\int((s : \mathbb{S}^1) \times C(s)) \simeq (u : \int \mathbb{S}^1) \times B\varphi(u)\]

and so an equivalence on their 0-truncations.
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• [RW] *Modal Descent*, Rijke, Wellen, TBD
• [CORS] *Localization in HoTT*, Christensen, Opie, Rijke, Scoccola, 2018
**Lemma**

If $X :: \text{Type}$ is *locally discrete* ($\mathfrak{f}$-separated), then for $x :: X$,

$$\text{BAut}_X(x) \equiv (y : X) \times \|x = y\|$$

is discrete.