Presheaves, Sets and Univalence

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Computer Science & Technology

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WANTED

More models of

Univalent Type Theory

e.g. to better understand the phenomenon of univalence
and its relationship to higher inductive types and propositional resizing

WANTED

More, and simpler, models of

Univalent Type Theory

Cf. the set-theoretic model of extensional MLTT
(using everyone's grasp of naive set theory).

Given what we know about homotopy theory,
perhaps having a very simple model of UTT is a forlorn hope.

Constructive logic to the rescue?

WANTED

More, and <u>simpler</u>, models of

UTT

Martin-Löf Type Theory with a hierarchy of universes
$(\mathcal{U}_n \mid n \in \mathbb{N})$ [non-cumulative, Russell-style]
closed under Π, Σ, $+$, W, N_0, N_1 and
"typal" identity types Id that make all $\mathcal{U}_n$ univalent

computation rule for Id
may not hold definitionally,
but the corresponding identity type
is inhabited

for all $X, Y : \mathcal{U}_n$, the type
$\mathsf{Id}_{\mathcal{U}_n}(X, Y)$ in $\mathcal{U}_{n+1}$
and the type of equivalences $X \simeq Y$
in $\mathcal{U}_n$
are canonically equivalent

Two existing models of UTT

- **Voevodsky *et al***:
  Simplicial sets with Kan-filling property
  within classical set theory with enough inaccessible cardinals.
- **Coquand *et al***:
  Cubical sets with uniform-Kan-filling structure (various flavours)
  within intuitionistic set theory with enough Grothendieck universes.

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Both models are constructed starting from presheaf categories $\mathbf{Set}^{\mathrm{C}^{\mathrm{op}}}$ for particular small categories C.

Which C are suitable?

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Both models are constructed starting from presheaf categories $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ for particular small categories $\mathbf{C}$.

Which $\mathbf{C}$ are suitable?

Idea: if intuitionistic set theory is the meta-theory
then maybe $\mathbf{C}$ can be 1
i.e. can we construct models in $\mathbf{Set}$ ($\cong \mathbf{Set}^{1^{\mathrm{op}}}$)?

[Historical aside]

Q Are there non-trivial sets X with $X \cong X^X$?

[$\rightsquigarrow$ straightforward models of untyped λ-calculus]

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Q Are there set-theoretic models of the polymorphic λ-calculus?

A ($\neg$Reynolds, AMP): yes, provided one works in intuitionistic rather than classical logic.

[Indeed, intuitionistically there are even small, non-posetal and yet complete categories (Moggi-Hyland-Robinson-Rosolini).]

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i.e. can we construct models in $\mathbf{Set}$ ($\cong \mathbf{Set}^{\mathbf{1}^{\mathrm{op}}}$)?

IZFA

Intuitionistic **Z**ermelo-**F**raenkel set theory with **A**toms

is a theory in intuitionistic first-order logic
(single-sorted, with equality)

so, no Law of Excluded Middle
$\varphi \vee \neg\varphi$

whose signature contains

one constant
$\boxed{\mathtt{I}}$ ("the set of atoms")

one binary relation
$\boxed{x \in X}$ ("x is an element of the set X")

and the following axioms…

[written using some definitional extensions]

IZFA axioms

$x \in \mathcal{S} \;\triangleq\; \neg(x \in \mathtt{I})$
means "x is a set"

so if x is an atom, then
for all y, $\neg(y \in x)$

- **Decidability:** $\forall x,\ x \in \mathtt{I} \vee x \in \mathcal{S}$
- **Sets:** $\forall X,\ (\exists x,\ x \in X) \Rightarrow X \in \mathcal{S}$
- **Extensionality:** $\forall X, Y \in \mathcal{S}, (\forall x,\ x \in X \Leftrightarrow x \in Y) \Rightarrow X = Y$
- **Induction:** $(\forall X, (\forall x \in X,\ \varphi(x)) \Rightarrow \varphi(X)) \Rightarrow \forall X,\ \varphi(X)$
- various axioms asserting the existence of sets:
  Separation
  Collection
  Unordered pairs
  Union
  Powerset
  Infinity

IZFA+ axioms

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- various axioms asserting the existence of sets:
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  Infinity
- a countable hierarchy $\mathcal{S}_0 \in \mathcal{S}_1 \in \mathcal{S}_2 \in \cdots \in \mathcal{S}$ of Grothendieck universes (transitive sets of sets closed under the above set-forming operations)

Presheaf models of IZFA+

Mike Fourman, *Sheaf Models for Set Theory*, JPAA 19(1980)91–101.

Dana Scott, *The Presheaf Model for Set Theory*, hand-written note, 1980.

For any small category $\mathbf{C}$, since $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is a topos, it can soundly interpret (higher-order) intuitionistic logic. So we can look for models of the theory IZFA there.

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In ZFC + countably many inaccessible cardinals, starting with any presheaf $\mathtt{I} \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ for atoms, we get a model of IZFA+ by solving fixed point equations in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$

$$\boxed{\mathcal{S}_n \cong P_n(\mathtt{I} + \mathcal{S}_n)}$$

where $P_n : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is a suitable size-n powerobject functor

by taking the colimit of a suitably long ordinal-iteration of $P_n(\mathtt{I} + \_)$.

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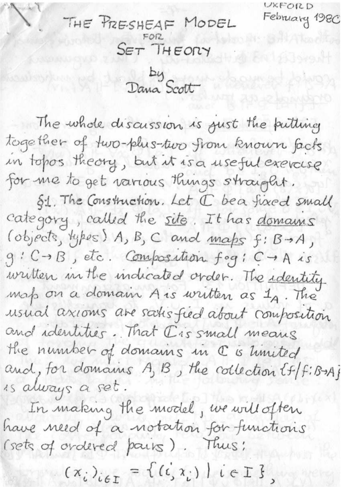

OXFORD
February 1980

THE PRESHEAF MODEL
FOR
SET THEORY
by
Dana Scott

The whole discussion is just the putting together of two-plus-two from known facts in topos theory, but it is a useful exercise for me to get various things straight.

§1. The Construction. Let $\mathbb{C}$ be a fixed small category, called the site. It has domains (objects, types) A, B, C and maps $f: B \to A$, $g: C \to B$, etc. Composition $f \circ g: C \to A$ is written in the indicated order. The identity map on a domain A is written as 1_A. The usual axioms are satisfied about composition and identities. That $\mathbb{C}$ is small means the number of domains in $\mathbb{C}$ is limited and, for domains A, B, the collection $\{f \mid f: B \to A\}$ is always a set.

In making the model, we will often have need of a notation for functions (sets of ordered pairs). Thus:

$$(x_i)_{i \in I} = \{(i, x_i) \mid i \in I\},$$

My take-home message from Scott's note:

- Each $\mathbf{C}$ determines a Kripke-like forcing interpretation of IZFA.
- Within that model of set theory there is a category of "global" sets and functions equivalent to $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ (and within that, a full subcategory equivalent to $\mathbf{C}$).
- So things that one might do concretely with presheaves (models of univalence!) can in principle be done in the model using the language of IZFA and might look simpler that way, because the set-theoretic interpretation of dependent types is simple…

Set-theoretic model of Type Theory

For the formalities see:

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From now on, let's work informally within IZFA+

- types are sets $X \in \mathcal{S}$
- dependent types are $\mathcal{S}$-valued functions, $F \in X \to \mathcal{S}$

$$
\begin{array}{rcl}
(x, y) & \triangleq & \{\{x\}, \{x, y\}\} \\
F \in X \to \mathcal{S} & \triangleq & \forall z \in F, \exists x \in X, \exists Y \in \mathcal{S},\ z = (x, Y)\ , \\
 & & \forall x, Y, Y',\ (x, Y), (x, Y') \in F \Rightarrow Y = Y'
\end{array}
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- dependent products $\Sigma\, X\, F \triangleq \{(x, y) \mid x \in X\,,\, y \in F(x)\}$

$$\boxed{y \in F(x) \;\triangleq\; \exists Y,\ (x, Y) \in F\,,\, y \in Y}$$

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- dependent functions $\Pi\, X\, F \triangleq$
  $\{f \in X \to \_ \mid \forall x \in X, \exists y \in F(x),\ (x, y) \in f\}$

$$\boxed{\begin{aligned} f \in X \to \_ \ \triangleq\ & \forall z \in f, \exists x, y,\ z = (x, y)\ , \\ & \forall x, y, y',\ (x, y), (x, y') \in f \Rightarrow y = y' \end{aligned}}$$

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$$\{f \in X \to \_ \mid \forall x \in X, \exists y \in F(x),\ (x, y) \in f\}$$

Use Agda-like notation $\boxed{\begin{array}{rcl} (x \in X) \times F(x) & \triangleq & \Sigma\, X\, F \\ (x \in X) \to F(x) & \triangleq & \Pi\, X\, F \end{array}}$

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  $\{f \in X \to \_ \mid \forall x \in X, \exists y \in F(x),\ (x, y) \in f\}$
- well-founded trees $\mathrm{W}\, X\, F \triangleq$ least W s.t.
  $(x \in X) \times (F(x) \to W) \subseteq W$

Martin-Löf Universes

Within IZFA, say that a set $\mathcal{U} \in \mathcal{S}$ is an MLU if

- $\mathcal{U} \subseteq \mathcal{S}$ (i.e. the elements of $\mathcal{U}$ are sets)
- $\emptyset$ and $\{\emptyset\}$ are in $\mathcal{U}$
- if $X, Y \in \mathcal{U}$ then
  $X \uplus Y \triangleq \{(\emptyset, x) \mid x \in X\} \cup \{(\{\emptyset\}, y) \mid y \in Y\}$ is in $\mathcal{U}$
- if $X \in \mathcal{U}$ and $F \in X \to \mathcal{U}$, then
  $\Sigma\, X\, F$, $\Pi\, X\, F$ and $\mathrm{W}\, X\, F$ are in $\mathcal{U}$

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- if $X \in \mathcal{U}$ and $F \in X \to \mathcal{U}$, then
  $\Sigma\, X\, F$, $\Pi\, X\, F$ and $\mathrm{W}\, X\, F$ are in $\mathcal{U}$

$\boxed{\text{What does it mean for such a } \mathcal{U} \text{ to be \underline{univalent}?}}$

Since univalence depends on identity, to answer the question, we first have to consider (typal) identity sets in an MLU…

Typal identity sets

To model typal identity types in IZFA (naively), we need for each $X \in \mathcal{S}$

- $\mathsf{Id}_X \in X \times X \to \mathcal{S}$
- $\mathsf{r}_X \in (x \in X) \to \mathsf{Id}_X(x, x)$

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- $\mathsf{elim}_X \in (x \in X) \to$
$$\begin{aligned}&(F \in (y \in X) \times \mathsf{Id}_X(x, y) \to \mathcal{S}) \to \\ &(z \in F(x, \mathsf{r}_X\, x)) \to \\ &(y \in X) \to \\ &(p \in \mathsf{Id}_X(x, y)) \to \\ &F(y, p)\end{aligned}$$
- $\mathsf{comp}_X \in (x \in X) \to$
$$\begin{aligned}&(F \in (y \in X) \times \mathsf{Id}_X(x, y) \to \mathcal{S}) \to \\ &(z \in F(x, \mathsf{r}_X\, x)) \to \\ &\mathsf{Id}_{F(x, \mathsf{r}_X\, x)}(\mathsf{elim}_X x\, F\, z\, x\, (\mathsf{r}_X\, x)\,,\, z)\end{aligned}$$

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  $(F \in (y \in X) \times \mathsf{Id}_X(x,y) \to \mathcal{S}) \to$
  $(z \in F(x, \mathsf{r}_X\, x)) \to$
  $(y \in X) \to$
  $(p \in \mathsf{Id}_X(x,y)) \to$
  $F(y,p)$

Given $x \in X$, taking $F_x(y,p) \triangleq \{\emptyset \mid x = y\,, p = \mathsf{r}_X\, x\}$,
for all $y \in X$ and $p \in \mathsf{Id}_X(x,y)$ we get $\mathsf{elim}_X\, x\, F_x\, \emptyset\, y\, p \in F_x(y,p)$
and hence $y = x$ and $p = \mathsf{r}_X\, x$. So $\boxed{\mathsf{Id}_X(x,y) \cong \{\emptyset \mid x = y\}}$
Therefore this typal identity coincides with extensional equality
and so univalence with respect to Id implies degeneracy.

Typal identity sets

To model typal identity types inside some MLUs $\mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2 \in \cdots$, we need for each $X \in \mathcal{U}_n$

- $\mathsf{Id}_X \in X \times X \to \mathcal{U}_n$
- $\mathsf{r}_X \in (x \in X) \to \mathsf{Id}_X(x, x)$
- $\mathsf{elim}_X \in (x \in X) \to$
$$\begin{aligned}&(F \in (y \in X) \times \mathsf{Id}_X(x, y) \to \mathcal{U}_m) \to\\ &(z \in F(x, \mathsf{r}_X\, x)) \to\\ &(y \in X) \to\\ &(p \in \mathsf{Id}_X(x, y)) \to\\ &F(y, p)\end{aligned}$$
- $\mathsf{comp}_X \in (x \in X) \to$
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How to get such structure? One way is via an interval…

Path sets

Extend IZFA+ by endowing $\mathtt{I}$ (the set of atoms) with some structure that makes it interval-like.

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Assuming constants $\mathtt{0}, \mathtt{1} \in \mathtt{I}$ (end points), for each $X \in \mathcal{S}$ and $x, y \in X$ we can define $\mathsf{Id}_X(x, y) \triangleq (x \sim y)$ where

$$\boxed{\begin{array}{rcl} x \sim y & \triangleq & \{p \in \mathtt{I} \to X \mid p\,\mathtt{0} = x\,,\, p\,\mathtt{1} = y\} \\ \mathsf{r}_X\, x & \triangleq & \{(i, x) \mid i \in \mathtt{I}\} \in (x \sim x) \end{array}}$$

What suffices for this to give typal identity types in the MLUs $\mathcal{U}_n$?

Path sets

$\mathtt{I}$-connection $\boxed{\begin{array}{c} \sqcap \in (\mathtt{I} \times \mathtt{I} \to \mathtt{I}) \text{ satisfying} \\ \mathtt{0} \sqcap i = \mathtt{0} = i \sqcap \mathtt{0} \\ \mathtt{1} \sqcap i = i = i \sqcap \mathtt{1} \end{array}}$

$\mathtt{I}$-coercion $\boxed{\mathsf{coe} \in ((P \in \mathtt{I} \to \mathcal{U}_n) \to P\,\mathtt{0} \to P\,\mathtt{1}) \text{ [no conditions]}}$

Suppose MLUs $\mathcal{U}_0 \in \mathcal{U}_1 \in \cdots$ are closed under $\mathtt{I}$-path sets:

$$\boxed{\forall X \in \mathcal{U}_n, \forall x, y \in X,\ (x \sim y) \in \mathcal{U}_n}$$

Theorem. If there is an $\mathtt{I}$-connection and an $\mathtt{I}$-coercion, then there exist elim and comp making $\sim$ a typal identity for the $\mathcal{U}_n$.

Sketch of the proof of the Theorem.

Adapting an argument due to Peter Lumsdaine [unpublished], it is possible to define a new version of $\mathtt{I}$-coercion

$$\overline{\mathsf{coe}} \in ((P \in \mathtt{I} \to \mathcal{U}_n) \to P\,\mathtt{0} \to P\,\mathtt{1})$$

which is "regular", i.e. for which there exist paths

$$\boxed{\overline{\mathsf{coe}}_\beta(X, x) \in \overline{\mathsf{coe}}\,(\lambda\_ \to X)\,x \sim x}$$

for all $x \in X \in \mathcal{U}_n$.

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for all $x \in X \in \mathcal{U}_n$.

Then given
$F \in ((y \in X) \times \mathsf{Id}_X(x, y) \to \mathcal{U}_m)$, $z \in F(x, \mathsf{r}_X\,x)$, $y \in X$, $p \in (x \sim y)$
using $\sqcap$, $\overline{\mathsf{coe}}$ and $\overline{\mathsf{coe}}_\beta$, one can define (following Martin-Löf)

$$\mathsf{elim}_X\,x\,F\,z\,y\,p \in F(y, p)$$
$$\mathsf{elim}_X\,x\,F\,z\,y\,p \triangleq \overline{\mathsf{coe}}(\lambda i \to F(p\,i, \lambda j \to p(i \sqcap j)))\,z$$

$$\mathsf{comp}_X\,x\,F\,z \in \mathsf{elim}_X x\,F\,z\,x\,(\mathsf{r}_X\,x) \sim z$$
$$\mathsf{comp}_X\,x\,F\,z \triangleq \overline{\mathsf{coe}}_\beta(F(x, \mathsf{r}_X\,x), z)$$

$\square$

Univalence

Suppose MLUs $\mathcal{U}_0 \in \mathcal{U}_1 \in \cdots$ are closed under $\mathtt{I}$-path sets, that $\mathtt{I}$ has a connection and that there are $\mathtt{I}$-coercions in $\mathcal{U}\_$, so that paths give typal identity types $\sim$ in $\mathcal{U}\_$.

Voevodsky's definition: $\mathcal{U}_n$ is univalent if for all $X, Y \in \mathcal{U}_n$, the canonical function (in $\mathcal{U}_{n+1}$)

$$(X \sim Y) \to (X \simeq Y)$$

is an equivalence

usual type of equivalences (mod $\sim$) in $\mathcal{U}_n$

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Observations of Shulman, Licata and specifically

Ian Orton and AMP, *Decomposing the Univalence Axiom*, TYPES 2017

leads to a simpler criterion for univalence…

Univalence

Theorem. The MLUs $\mathcal{U}_\_$ are univalent iff there are elements

$$\begin{aligned}
\mathrm{uc} &\in (X \in \mathcal{U}_n) \to \mathrm{isContr}(X) \to (X \sim 1) \\
\mathrm{ub} &\in (X, Y \in \mathcal{U}_n) \to (X \cong Y) \to (X \sim Y) \\
\mathrm{ub}_\beta &\in (X, Y \in \mathcal{U}_n)(b \in X \cong Y)(x \in X) \to \\
&\qquad \overline{\mathrm{coe}}(\mathrm{ub}\, X\, Y\, b)\, x \sim b(x)
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$(x \in X) \times (y \in X) \to x \sim y$

set-theoretic bijections

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\end{aligned}$$

Proof uses fact that functions are extensional mod $\sim$:

$$\begin{aligned}
\mathsf{funext} &\in (e \in (x \in X) \to (f\,x \sim g\,x)) \to (f \sim g) \\
\mathsf{funext}\, e\, i &\triangleq \lambda(x \in X) \to e\, x\, i
\end{aligned}$$

Instead of Conclusions, I have some questions

Question 1

? Are there any models of IZFA+ for which

- $\mathtt{I}$ has a connection $\sqcap$ and is non-trivial ($\mathtt{0} \neq \mathtt{1}$)
- there are MLUs $\mathcal{U}_0 \in \mathcal{U}_1 \in \ldots$ closed under $\mathtt{I}$-path sets, with $\mathtt{I}$-coercions coe and a univalence structure $\mathrm{uc}, \mathrm{ub}, \mathrm{ub}_\beta$?

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In the model of IZFA+ using the presheaf topos from

[CCHM] C. Cohen, T. Coquand, S. Huber and A. Mörtberg, *Cubical type theory: a constructive interpretation of the univalence axiom* [arXiv:1611.02108]

one can get a "Tarski" version of the above…

Tarski-style

$$\boxed{\mathsf{E} \in \mathcal{U}_n \to \mathcal{S}}$$

- $\hat{\mathsf{Id}} \in (X \in \mathcal{U}_n) \to \mathsf{E}\,X \to \mathsf{E}\,X \to \mathcal{U}_n$ satisfying $\mathsf{E}(\hat{\mathsf{Id}}\,X\,x\,y) = (x \sim y)$ and similarly for $\hat{0}$, $\hat{1}$, $\hat{+}$, $\hat{\Pi}$, $\hat{\Sigma}$ and $\hat{\mathsf{W}}$
- $\hat{\mathcal{U}}_n \in \mathcal{U}_{n+1}$ satisfying $\mathsf{E}(\hat{\mathcal{U}}_n) = \mathcal{U}_n$
- $\widehat{\mathsf{coe}} \in (P \in \mathbb{I} \to \mathcal{U}_n) \to \mathsf{E}(P\,0) \to \mathsf{E}(P\,1)$
- $\widehat{\mathsf{uc}} \in (X \in \mathcal{U}_n) \to \mathsf{isContr}(\mathsf{E}\,X) \to (X \sim \hat{1})$
  $\widehat{\mathsf{ub}} \in \cdots$
  $\widehat{\mathsf{ub}}_\beta \in \cdots$

In the CCHM model of IZFA+, $\mathsf{E} : \mathcal{U}_n \to \mathcal{S}$ is the carrier for the generic CCHM fibration (with size-n fibres)

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one can get a "Tarski" version of the above…
Can this be "Russellified"?

Question 2

D. R. Licata, I. Orton, AMP and B. Spitters, *Internal Universes in Models of Homotopy Type Theory*, in FSCD 2018.

constructs the [CCHM] univalent universe entirely within dependent type theory + a local/global modality ("Crisp" Type Theory), starting from axiom

the interval is "tiny"

(i.e. $(\_)^{\mathbb{I}}$ has a global right adjoint)

plus the Orton-Pitts axioms for the interval and cofibrant propositions.

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? Is there a modal version of IZFA+ admitting a set-theoretic model of Crisp Type Theory?

Question 3

B. van den Berg and I. Moerdijk, *Univalent Completion*,
Math. Ann. 371(2018)1337–1350 [doi:10.1007/s00208-017-1614-3]

Is there a generalisation of the result about classical Kan simplicial sets in the above paper?

? In IZFA+, can any MLUs $\mathcal{U}_0 \in \mathcal{U}_1 \in \ldots$ that are closed under $\mathbb{I}$-path sets and with $\mathbb{I}$-coercions be completed to MLUs with a univalence structure?

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Thanks for your attention — any answers?