Homotopy canonicity of homotopy type theory

Christian Sattler
jww. Chris Kapulkin

University of Nottingham

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Introduction: constructivity of MLTT

Martin-Löf type theory (MLTT) is a constructive system:

- existence proofs are effective,
- can be used as programming language with notion of evaluation.

It enjoys **canonicity**: every closed term of a positive type (e.g., Nat) is obtained (up to judgmental equality) from an introduction rule.

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Adding axioms (e.g., law of excluded middle) destroys canonicity.

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Conjecture (Voevodsky, ≤ 2010[1])

For any closed term n of natural number type, there is $k \in \mathbb{N}$ with a closed term p of the identity type relating n to the numeral $S^k 0$. Both n and p may make use of the univalence axiom.

Furthermore, this procedure should be given by an effective algorithm.

This is known as the **homotopy canonicity conjecture**.

[1] Vladimir Voevodsky, Univalent Foundations Project, `http://www.math.ias.edu/vladimir/files/univalent_foundations_project.pdf`

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Furthermore, this procedure should be given by an effective algorithm.

This is known as the **homotopy canonicity conjecture**.

"This conjecture seems to be highly non-trivial. [...] I find this conjecture to be both very important for the univalent foundations and very interesting."

---

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Introduction: cubical type theories

Cubical type theories are extensions or modifications of HoTT with

- strict cubical shapes,
- additional operations and judgmental equations

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Important developments, but:

- Unknown if cubical type theories are conservative over HoTT. So this does **not** solve the homotopy canonicity conjecture.
- (Unclear if there is a cubical type theory that can be interpreted in standard homotopy types, let alone higher topoi.)
- (Strict cubical shapes are in contrast to weak axiomatization of higher groupoidal structure encoded intrinsically by identity type.)

MLTT: semantics

By MLTT, we understand the following collection of type formers:

- dependent sums (strict),
- dependent products (strict),
- indexed inductive types (in particular: identity types),
- hierarchy of universes (can be cumulative) closed under type formers.

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Write MLTT also for category of models[2] of this theory. A model $\mathcal{C}$ has:

- a category $\mathcal{C}$ of contexts and substitutions,
- presheaves $\mathsf{Ty} \in \widehat{\mathcal{C}}$ of types and $\mathsf{Tm} \in \widehat{\int \mathsf{Ty}}$ of terms,
- interpretations of global context 1 and context extension $\Gamma.A$,
- interpretations of above type formers, stable under substitution.

For example, we have the set model $\mathbf{Set} \in \mathsf{MLTT}$.

[2] presented using categories with families, categories with attributes, full comprehension categories, natural models, or any other equivalent notion

MLTT: canonicity (recollection)

Canonicity is a property of the initial model $0_{\mathsf{MLTT}} \in \mathsf{MLTT}$.

Canonicity is proved abstractly by sconing (Freyd cover), i.e. **glueing** along the global sections functor to **Set**.

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Glueing makes sense generally along a **pseudomorphism** of models:

Definition

A *pseudomorphism* $F : \mathcal{C} \to \mathcal{D}$ of models of MLTT is a functor on underlying categories with natural transformations on types and terms that preserves the global context and context extension up to (canonical) isomorphism and preserves small types (elements of universes).

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This is the analogue of a left exact functor.

Note: a pseudomorphism is **not** a morphism in MLTT.

MLTT: glueing (recollection)

Construction

Let $F : \mathcal{C} \to \mathcal{D}$ be a pseudomorphism of models of MLTT.
*The **glueing** $\mathrm{Glue}(F)$ is a model with category of contexts $\mathcal{D} \downarrow F$.*
The projection $\mathcal{D} \downarrow F \to \mathcal{C}$ extends to a map $\mathrm{Glue}(F) \to \mathcal{C}$ in MLTT.

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Construction

Let $F \colon \mathcal{C} \to \mathcal{D}$ *be a pseudomorphism of models of MLTT.*
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The projection $\mathcal{D} \downarrow F \to \mathcal{C}$ *extends to a map* $\mathsf{Glue}(F) \to \mathcal{C}$ *in* MLTT*.*

Concretely:

- A context is a triple of $\Gamma_{\mathcal{C}} \in \mathcal{C}$, $\Gamma_{\mathcal{D}} \in \mathcal{D}$, and $\Gamma_{\mathcal{D}} \xrightarrow{\gamma} F\Gamma_{\mathcal{C}}$.
- A types is a pair $A_{\mathcal{C}} \in \mathsf{Ty}_{\mathcal{C}}(\Gamma_{\mathcal{C}})$ and $A_{\mathcal{D}} \in \mathsf{Ty}_{\mathcal{D}}(\Gamma_{\mathcal{D}}.(FA_{\mathcal{C}})\sigma)$.
- A term is a pair $t_{\mathcal{C}} \in \mathsf{Tm}_{\mathcal{C}}(\Gamma_{\mathcal{C}}, A_{\mathcal{C}})$ and $t_{\mathcal{D}} \in \mathsf{Tm}_{\mathcal{D}}(\Gamma_{\mathcal{D}}, A_{\mathcal{D}}[(Ft_{\mathcal{C}})\gamma])$.
- Dependent sums, dependent products, and indexed inductive types are defined from the corresponding type formers in $\mathcal{C}$ and $\mathcal{D}$.
- Universes are interpreted using universes in $\mathcal{C}$ and $\mathcal{D}$ and non-dependent products in $\mathcal{D}$:

$$\mathsf{U}_{\mathsf{Glue}(F)} = (\mathsf{U}_{\mathcal{C}}, (F\,\mathsf{El}_{\mathcal{C}})\gamma \to \mathsf{U}_{\mathcal{D}}),$$
$$\mathsf{El}_{\mathsf{Glue}(F)}(A_{\mathcal{C}}, A_{\mathcal{D}}) = (\mathsf{El}_{\mathcal{C}}(A_{\mathcal{C}}), El_{\mathcal{D}}(\mathsf{app}(A_{\mathcal{D}}, \mathsf{q}))).$$

MLTT: canonicity from glueing (recollection)

The global sections functor $F\colon 0_{\mathsf{MLTT}} \to \mathbf{Set}$ extends to a pseudomorphism of models.

By initiality of 0_{MLTT}, we obtain a unique section to the projection from the glueing along F:

$$
\begin{tikzcd}
 & \mathrm{Glue}(F) \arrow[d] \\
0_{\mathsf{MLTT}} \arrow[ur, dotted, bend left, "{[\![-]\!]}", "\exists!"'] \arrow[r, "\mathrm{id}"] & 0_{\mathsf{MLTT}}
\end{tikzcd}
\qquad \text{(diagram in MLTT).}
$$

MLTT: canonicity from glueing (recollection)

By construction of $\mathrm{Glue}(F)$, we have $\mathrm{Nat}_{\mathrm{Glue}(F)} = (\mathrm{Nat}, \mathrm{Nat}')$ where $\mathrm{Nat}' \colon \mathrm{Tm}(1, \mathrm{Nat}) \to \mathbf{Set}$ is inductively generated by

$$
\begin{aligned}
0' &\in \mathrm{Nat}'(F0), \\
S'(m, m') &\in \mathrm{Nat}'((FS)(m)) \text{ for } m' \in \mathrm{Nat}'(m).
\end{aligned}
$$

This is the preimage of the canonical map $S^{(-)}0 \colon \mathbb{N} \to \mathrm{Tm}(1, \mathrm{Nat})$.

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This is the preimage of the canonical map $S^{(-)}0 \colon \mathbb{N} \to \mathrm{Tm}(1, \mathrm{Nat})$.

For canonicity, let $n \in \mathrm{Tm}_{0_{\mathbf{MLTT}}}(1, \mathrm{Nat})$.

We obtain $[\![n]\!] = (n, n') \in \mathrm{Tm}_{\mathrm{Glue}(F)}(1, \mathrm{Nat})$ where $n' \in \mathrm{Nat}'(n)$ is the witness that n is canonical (equal to a numeral $S^k 0$).

Semantics of HoTT

By HoTT, we mean the extension of MLTT with:

- witnesses of function extensionality (substitution stable),
- witnesses of univalence (substitution stable).

Everything we will say also applies when one adds:

- some higher inductive types such as pushouts,
- witnesses of resizing axioms (substitution stable).

Again, write HoTT also for category of models of this theory.

Shulman's work on homotopy canonicity

(Shulman; 2013)[3] generalized glueing along a pseudomorphism F to HoTT under the additional assumption that F coherently preserves **anodyne maps** (behaving as reflexivity for identity type elimination).

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Proposition (Shulman)

Given a pseudomorphism $F\colon \mathcal{C} \to \mathcal{D}$ *of models of* HoTT, *the glueing* $\mathrm{Glue}(F)$ *exists as soon as* F *preserves contractibility (stably under substitution).*

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This aligns the construction with Artin glueing for higher toposes: left exact functors are presented already by maps of fibration categories.

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Shulman: 0-truncated homotopy canonicity

The proof of canonicity of MLTT fails for HoTT for two reasons:

- **Set** is not a model of HoTT.
- The global sections functor $F : 0_{\mathsf{HoTT}} \to \mathbf{Set}$ does not preserve contractibility.

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Let n-**truncated** $HoTT^{\leq n}$ be the following modification of HoTT:

- the axiom K_n that every type is n-truncated (substitution stable),
- additional universes of $(n-1)$-types,
- univalence restricted to these universes.

Then **Set** is a model of 0-truncated HoTT.

Shulman: 0-truncated homotopy canonicity

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The global sections functor $0_{\mathsf{HoTT}^{\leq 0}} \to \mathbf{Set}$ fails to preserve contractibility.

Shulman: consider quotiented global sections functor $0_{\mathsf{HoTT}^{\leq 0}} \to \mathbf{Set}$ that sends a context Γ to the quotient $0_{\mathsf{HoTT}^{\leq 0}}(1, \Gamma)/\sim$ where $\gamma \sim \gamma'$ if the identity context $\mathsf{Id}_\Gamma(\gamma, \gamma')$ has a point.

Now the proof by sconing goes through. Instead of canonicity, we obtain **homotopy canonicity** for 0-truncated HoTT.

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Shulman: we obtain homotopy canonicity of 1-truncated HoTT by

- replacing **Set** with the model in **Gpd** (Hofmann, Streicher),
- using a groupoid-valued global sections functor $F\colon 0_{\mathrm{HoTT}^{\leq 0}} \to \mathbf{Gpd}$, quotiented on the level of morphisms; for closed type A:
  - the objects of FA are $\mathrm{Tm}(1, A)$,
  - the morphisms $FA(s, t)$ are $\mathrm{Tm}(1, \mathrm{Id}_A(s, t))/\sim$ where $u \sim v$ if $\mathrm{Tm}(1, \mathrm{Id}_{\mathrm{Id}_A(s,t)}(u, v))$ is inhabited.

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Note: the quotient on the level of morphisms is essential in two ways:

- to ensure preservation of contractibility,
- to make F a functor (instead of a pseudofunctor).

Shulman: problems in higher dimensions

What about higher dimensions?

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- Can define semisimplical structure:

$$\begin{aligned}(FA)_0 &= \mathsf{Tm}(1, A)\\ (FA)_1(s,t) &= \mathsf{Tm}(1, \mathsf{Id}_A(s,t))\\ (FA)_2(\ldots) &= \{\text{triangle fillers}\}\\ &\ldots\end{aligned}$$

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- However, we have no *natural* way to define degeneracies.
- But the real problem is that $(FA)_1$ is not functorial in A. This would mean $\mathsf{ap}(g \circ f) = \mathsf{ap}(g) \circ \mathsf{ap}(f)$, which only holds up to identity type.

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Brainstorming:

- Stop trying to glue the initial model.
- Types need to come functorially with data that encodes their higher-dimensional groupoidal structure.
- How do we do this without losing the connection to the initial model?

Shulman: inverse diagram models

Let D be a direct category with finite slices.

Construction (Shulman)

Any model $\mathcal{E} \in \mathrm{HoTT}$ *lifts to a model* $[D^{\mathrm{op}}, \mathcal{E}]_{\mathrm{rf}} \in \mathrm{HoTT}$ *of* ***inverse diagrams*** *in* $\mathcal{E}$ *over* D^{op}*.*

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*Any model $\mathcal{E} \in$ HoTT lifts to a model $[D^{\mathrm{op}}, \mathcal{E}]_{\mathrm{rf}} \in$ HoTT of **inverse diagrams** in $\mathcal{E}$ over D^{op}.*

Concretely:

- The contexts are functors $D^{\mathrm{op}} \to \mathcal{E}$.
- $A \in \mathrm{Ty}(\Gamma)$ is a structured **Reedy fibration**: $A_i \in \mathrm{Ty}(\Gamma_i.M_i A)$ for $i \in D$ where $M_i A$ is the telescope

$$M_i A = (x_j : \Gamma_f^* A_j[(\Gamma_g^* x_k)_{k \xrightarrow[\neq]{g} j}])_{j \xrightarrow[\neq]{f} i}.$$

- $t \in \mathrm{Tm}(\Gamma, A)$ consists of $\mathrm{Tm}(\Gamma_i, A_i[(\Gamma_f^* t_j)_{j \xrightarrow[\neq]{f} i}])$.
- ...

Variation: homotopical inverse diagram models

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Construction

Assume that for all $i \in D$, the weak equivalences with target i are either all maps or only the identity on i.

Then any model $\mathcal{E} \in \mathrm{HoTT}$ lifts to a model $[(D, W)^{\mathrm{op}}, \mathcal{E}]_{\mathrm{rf}} \in \mathrm{HoTT}$ of ***homotopical inverse diagrams*** *in $\mathcal{E}$ over $(D, W)^{\mathrm{op}}$.*

Variation: homotopical inverse diagram models

Concretely:

- Basic structure of $[(D, W)^{\mathrm{op}}, \mathcal{E}]_{\mathrm{rf}}$ is as in $[D^{\mathrm{op}}, \mathcal{E}]_{\mathrm{rf}}$, but with extra data on types $A \in \mathrm{Ty}(\Gamma)$ that $(\Gamma.A)_i \to (\Gamma.A)_j$ is fiberwise equivalence over $\Gamma_i \to \Gamma_j$ for any weak equivalence $j \to i$.
- Type formers in $[(D, W)^{\mathrm{op}}, \mathcal{E}]_{\mathrm{rf}}$ are interpreted as in $[D^{\mathrm{op}}, \mathcal{E}]_{\mathrm{rf}}$, with the exception of universes.

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- Type formers in $[(D, W)^{\mathrm{op}}, \mathcal{E}]_{\mathrm{rf}}$ are interpreted as in $[D^{\mathrm{op}}, \mathcal{E}]_{\mathrm{rf}}$, with the exception of universes.

For positive dependent sums, identity types, and dependent product, this construction is (Kapulkin, Lumsdaine; 2018)[5].

For the special case of $D = \{0 \rightrightarrows 1\}$ with W maximal, this construction appears (obfuscated by syntax) in (Tabareau, Tanter, Sozeau; 2018)[6].

[5]Homotopical inverse diagrams in categories with attributes

[6]Equivalences for free: univalent parametricity for effective transport

Example: frame models

Take $D = \Delta_+$ (the semisimplex category) and W maximal.

For $\mathcal{E} \in \mathsf{HoTT}$, yields model $[(\Delta_+, W)^{\mathsf{op}}, \mathcal{E}]_{\mathsf{rf}} \in \mathsf{HoTT}$ of **frames** in $\mathcal{E}$.

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These are the frames from Karol's talk yesterday:

- used for homotopical models of HoTT by (Kapulkin, Szumiło; 2017)[7],
- introduced for fibration categories by (Schwede; 2013)[8],
- goes back to simplicial notions of frame in model categories.

[7] Internal Languages of Finitely Complete $(\infty, 1)$-categories

[8] The p-order of topological triangulated categories

Example: frame models

Evaluation $\mathsf{ev}_0 \colon [(\Delta_+, W)^{\mathsf{op}}, \mathcal{E}]_{\mathsf{rf}} \to \mathcal{E}$ at level 0 gives a map in HoTT that is a weak equivalence in a certain precise sense (cf. Karol's talk).

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This model supports a semisimplicial global sections functor

$$[(\Delta_+, W)^{\mathrm{op}}, \mathcal{E}]_{\mathrm{rf}} \xrightarrow{[(\Delta_+, W)^{\mathrm{op}}, G]} \widehat{\Delta_+}$$

by applying the ordinary global sections functor G levelwise.

Homotopicality ensures that types are mapped to Kan fibrations in $\widehat{\Delta_+}$.

Glueing along semisimplicial global sections functor?

Unfortunately, semisimplicial sets do not model HoTT.

To fix this, one can weaken the η-rule for dependent products and relax some substitutional coherence (similar sacrifices are made in (Gambino, Henry; 2019)[9]).

Unfortunately, strict dependent products (i.e., with η) are needed for the construction of universes in inverse diagram models.

[9] Towards a constructive simplicial model of Univalent Foundations, 2019

Glueing along what?

Last piece of the puzzle: postcompose with **right Kan extension** $i_* \colon \widehat{\Delta_+} \to \widehat{\Delta}$ to move to simplicial sets (where $i \colon \Delta_+ \hookrightarrow \Delta$)!

Key: i_* is an exact functor of fibration categories[10,11], so it preserves types and contractibility.

[10] Henry, Weak model categories in classical and constructive mathematics, 2018

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In total, the functor $F_{\mathcal{E}}$ to glue along is (restricting to contextual $\mathcal{E}$):

$$[(\Delta_+, W)^{\mathrm{op}}, \mathcal{E}] \xrightarrow{[(\Delta_+, W)^{\mathrm{op}}, G]} (\widehat{\Delta_+})_{\mathsf{fib}} \xrightarrow{\quad i_* \quad} (\widehat{\Delta})_{\mathsf{fib}}.$$

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Remaining problems: substitutional coherence

We did not explain how to achieve an action of F on (contractible) types that is coherent with substitution.

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We solve this problem by:

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- replacing Voevodsky's big universe in $\widehat{\Delta}$ to define types by a Hofmann-Streicher one.

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In simplicial sets, define

$$\mathrm{U}_{\mathrm{big},n} = \{X \in \widehat{\Delta/[n]} \mid \bullet \xrightarrow{X} \Delta^n \text{ Kan fibration}\},$$
$$\mathrm{U}^{\leq -2}_{\mathrm{big},n} = \{X \in \widehat{\Delta/[n]} \mid \bullet \xrightarrow{X} \Delta^n \text{ trivial fibration}\}.$$

Note that the universal fibration of $\mathrm{U}^{\leq -2}_{\mathrm{big},n}$ is a contractible type.

We then have natural transformations

$$\mathrm{Ty} \to \widehat{\Delta}(F_{\mathcal{E}}(-), \mathrm{U}_{\mathrm{big}}),$$
$$\mathrm{Ty}^{\leq -2} \to \widehat{\Delta}(F_{\mathcal{E}}(-), \mathrm{U}^{\leq -2}_{\mathrm{big}}).$$

Homotopy canonicity

Now we can glue.

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To obtain homotopy of HoTT, we use initiality of 0_{HoTT}:

$$\begin{tikzcd}
 & \mathrm{Glue}(F_{0_{\mathsf{HoTT}}}) \arrow[d] \\
 & {[(\Delta_+, W)^{\mathrm{op}}, 0_{\mathsf{HoTT}}]} \arrow[d, "\mathsf{ev}_{\mathbf{0}}"] \\
0_{\mathsf{HoTT}} \arrow[r, "\mathrm{id}"'] \arrow[ur, dotted, bend left, "{[\![-]\!]}"] \arrow[uur, dotted, bend left, "{[\![-]\!]}"] & 0_{\mathsf{HoTT}}
\end{tikzcd}$$

(diagram in HoTT).

Homotopy canonicity

Now we can glue.

To obtain homotopy of HoTT, we use initiality of 0_{HoTT}:

$$\begin{tikzcd}
 & \mathsf{Glue}(F_{0_{\mathsf{HoTT}}}) \arrow[d] \\
 & {[(\Delta_+, W)^{\mathrm{op}}, 0_{\mathsf{HoTT}}]} \arrow[d, "\mathsf{ev}_0"] \\
0_{\mathsf{HoTT}} \arrow[r, "\mathsf{id}"'] \arrow[uur, dotted, bend left, "{[\![-]\!]}"] \arrow[ur, dotted, bend left, "{[\![-]\!]}"'] & 0_{\mathsf{HoTT}}
\end{tikzcd} \qquad \text{(diagram in HoTT).}$$

Note that the natural numbers are interpreted in $\widehat{\Delta}$ as constantly $\mathbb{N}$.

Given $n \in \mathsf{Tm}(1, \mathsf{Nat})$, the second component of $[\![n]\!]$ in $\mathsf{Glue}(F_{0_{\mathsf{HoTT}}})$ gives us $k \in \mathbb{N}$ with $\mathsf{Tm}(1, \mathsf{Id}_{\mathsf{Nat}}([\![n]\!], S^k 0))$ in $[(\Delta_+, W)^{\mathrm{op}}, 0_{\mathsf{HoTT}}]$.

Evaluating at level 0, we obtain $\mathsf{Tm}(1, \mathsf{Id}_{\mathsf{Nat}}(n, S^k 0))$ in 0_{HoTT}.

Constructivity issues

So far, the proof of homotopy canonicity is non-constructive because the simplicial set model (with the needed strictness) is non-constructive.

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Bezem: we still obtain a (very stupid) terminating algorithm.

Can we do better?

Homotopy canonicity via glueing with cubical sets

We obtain a constructive proof of homotopy canonicity by glueing along:

$$\begin{array}{ll}
[(\Delta_+, W)^{\mathrm{op}}, 0_{\mathsf{HoTT}}] & \\
\big\downarrow {\scriptstyle [(\Delta_+, W)^{\mathrm{op}}, G]} & \\
(\widehat{\Delta_+})_{\mathsf{dec,fib}} & \text{(levelwise decidable equality)} \\
\big\downarrow {\scriptstyle i_*} & \\
\widehat{\Delta}_{\mathsf{degunifib}} & \text{(degeneracy-uniform fibrations)} \\
\big\downarrow {\scriptstyle \mathsf{Id}} & \\
\widehat{\Delta}_{\mathsf{unifib}} & \text{(uniform fibrations)} \\
\big\downarrow {\scriptstyle j_* \text{ where } j \colon \Delta \to \mathsf{FinLat} = \overline{\square_{\mathbf{full}}}} & \\
(\widehat{\square_{\mathsf{full}}})_{\mathsf{CCHM}}. &
\end{array}$$

Homotopy canonicity via glueing with cubical sets

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\quad\big\downarrow{\scriptstyle j_* \text{ where } j\colon \Delta \to \mathsf{FinLat} = \overline{\square_{\mathbf{full}}}} & \\
(\widehat{\square_{\mathsf{full}}})_{\mathsf{CCHM}}. &
\end{array}
$$

The coherence issue of the action on (contractible) types is solved using the modification of the local universe method of (Shulman; 2019)[12].

[12] All $(\infty, 1)$-toposes have strict univalent universes