All $(\infty,1)$-toposes have strict univalent universes

Mike Shulman

University of San Diego

HoTT 2019
Carnegie Mellon University
August 13, 2019
A (Grothendieck–Rezk–Lurie) $(\infty, 1)$-topos is:

- The category of objects obtained by “homotopically gluing together” copies of some collection of “model objects” in specified ways.
- The free cocompletion of a small $(\infty, 1)$-category preserving certain well-behaved colimits.
- An accessible left exact localization of an $(\infty, 1)$-category of presheaves.

They are a powerful tool for studying all kinds of “geometry” (topological, algebraic, differential, cohesive, etc.).

It has long been expected that $(\infty, 1)$-toposes are models of HoTT, but coherence problems have proven difficult to overcome.
Theorem (S.)

Every $(\infty, 1)$-topos can be given the structure of a model of “Book” HoTT with strict univalent universes, closed under $\Sigma$s, $\Pi$s, coproducts, and identity types.

Caveats for experts:

1. Classical metatheory: ZFC with inaccessible cardinals.
2. We assume the initiality principle.
3. Only an interpretation, not an equivalence.
4. HITs also exist, but remains to show universes are closed under them.
Example

1. Hou–Finster–Licata–Lumsdaine formalized a proof of the Blakers–Massey theorem in HoTT.

2. Later, Rezk and Anel–Biedermann–Finster–Joyal unwound this manually into a new $(\infty, 1)$-topos-theoretic proof, with a generalization applicable to Goodwillie calculus.

3. We can now say that the HFLL proof already implies the $(\infty, 1)$-topos-theoretic result, without manual translation. (Modulo closure under HITs.)
Outline

1. Type-theoretic model toposes
2. Left exact localizations
3. Injective model structures
4. Remarks
We can interpret type theory in a well-behaved model category $\mathcal{E}$:

<table>
<thead>
<tr>
<th>Type theory</th>
<th>Model category</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type $\Gamma \vdash A$</td>
<td>Fibration $\Gamma.A \to \Gamma$</td>
</tr>
<tr>
<td>Term $\Gamma \vdash a : A$</td>
<td>Section $\Gamma \to \Gamma.A$ over $\Gamma$</td>
</tr>
<tr>
<td>Id-type</td>
<td>Path object</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>Universe</td>
<td>Generic small fibration $\pi : \tilde{U} \to U$</td>
</tr>
</tbody>
</table>

To ensure $U$ is closed under the type-forming operations, we choose it so that every fibration with “$\kappa$-small fibers” is a pullback of $\pi$, where $\kappa$ is some inaccessible cardinal.
Universes in presheaves

Let $\mathcal{E} = [\mathcal{C}^{\text{op}}, \text{Set}]$ be a presheaf model category.

**Definition**

Define a presheaf $U \in \mathcal{E} = [\mathcal{C}^{\text{op}}, \text{Set}]$ where

$$U(c) = \left\{ \kappa\text{-small fibrations over } \mathcal{C}(\mathcal{C}(-, c)) \right\}$$

with functorial action by pullback along $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$.

(Plus standard cleverness to make it strictly functorial.)

Similarly, define $\tilde{U}$ using fibrations equipped with a section. We have a $\kappa$-small map $\pi : \tilde{U} \to U$.

**Theorem**

*Every $\kappa$-small fibration is a pullback of $\pi$.*

But $\pi$ may not itself be a fibration!
Theorem

If the generating acyclic cofibrations in \( \mathcal{E} = [\mathcal{C}^{\text{op}}, \text{Set}] \) have representable codomains, then \( \pi : \tilde{U} \to U \) is a fibration.

Proof.

To lift in the outer rectangle, instead lift in the left square.

```
\begin{tikzcd}
A \rar & \bullet \rar & \tilde{U} \\
\sim \urar & x \ur & \downarrow \pi \\
\mathbb{A} \urar & \mathbb{A} \urar & [x] \rar & U
\end{tikzcd}
```

Example (Voevodsky)

In simplicial sets, the generating acyclic cofibrations are \( \Lambda^{n,k} \to \Delta^n \), where \( \Delta^n \) is representable.
In cubical sets, the fibrations have a *uniform choice* of liftings against generators $\Box^{n,k} \to \Box^n$. Since $\Box^n$ is representable, our $\pi$ lifts against these generators, but not uniformly.

Instead one defines (BCH, CCHM, ABCFHL, etc.)

$$U(c) = \left\{ \text{small fibrations over } \mathcal{C} \text{ with specified uniform lifts} \right\}.$$ 

Then the lifts against the generators $\Box^{n,k} \to \Box^n$ cohere under pullback, giving $\pi$ also a uniform choice of lifts.

Let’s put this in an abstract context.
Notions of fibred structure

Definition

A notion of fibred structure $\mathbb{F}$ on a category $\mathcal{E}$ assigns to each morphism $f : X \to Y$ a set (perhaps empty) of “$\mathbb{F}$-structures”, which vary functorially in pullback squares: given a pullback

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
$$

any $\mathbb{F}$-structure on $f$ induces one on $f'$, functorially.

Definition

A notion of fibred structure $\mathbb{F}$ is **locally representable** if for any $f : X \to Y$, the functor $\mathcal{E}/Y \to \text{Set}$, sending $g : Z \to Y$ to the set of $\mathbb{F}$-structures on $g^*X \to Z$, is representable.
Notions of fibration structure

Examples

The following notions of fibred structure on a map $f : X \to Y$ are locally representable:

1. The property of lifting against a set of maps with representable codomains (e.g. simplicial sets).
2. The *structure* of liftings against a *category* of maps with representable codomains (e.g. as in Emily’s talk).
3. A $G_Y$-algebra structure for a fibred pointed endofunctor $G$ (e.g. the partial map classifier, as in Steve’s talk).
4. A section of $F_Y(X)$, for any fibred endofunctor $F$.
5. The combination of two or more locally representable notions of fibred structure.
6. The property of having $\kappa$-small fibers.
7. A square exhibiting $f$ as a pullback of some $\pi : \widetilde{U} \to U$. 
Universes from fibration structures

For a notion of fibred structure $F$, define

$$U(c) = \left\{ \text{small maps into } \mathcal{F}c \text{ with specified } F\text{-structures} \right\}.$$

and similarly $\pi : \tilde{U} \to U$.

**Theorem**

*If $F$ is locally representable, then $\pi$ also has an $F$-structure, and every $F$-structured map is a pullback of it.*

**Proof.**

Write $U$ as a colimit of representables. All the coprojections factor coherently through the representing object for $F$-structures on $\pi$, so the latter has a section.

(Can also use the representing object for $F$-structures on the classifier $\tilde{V} \to V$ of all $\kappa$-small morphisms, as Steve did yesterday.)
Type-theoretic model toposes

Definition (S.)

A type-theoretic model topos is a model category $\mathcal{E}$ such that:

- $\mathcal{E}$ is a right proper Cisinski model category.
- $\mathcal{E}$ has a well-behaved, locally representable, notion of fibred structure $\mathbb{F}$ such that the maps admitting an $\mathbb{F}$-structure are precisely the fibrations.
- $\mathcal{E}$ has a well-behaved enrichment (e.g. over simplicial sets).

It is not hard to show:

1. Every type-theoretic model topos interprets Book HoTT with univalent universes. ($\text{FEP} + \text{EEP} \Rightarrow U$ is fibrant and univalent.)
2. The $(\infty, 1)$-category presented by a type-theoretic model topos is a Grothendieck $(\infty, 1)$-topos. (It satisfies Rezk descent.)

The hard part is the converse of (2): are there enough ttmts?
The Plan

An $(\infty, 1)$-topos is, by one definition, an accessible left exact localization of a presheaf $(\infty, 1)$-category. Thus it will suffice to:

1. Show that simplicial sets are a type-theoretic model topos. ✓
2. Show that type-theoretic model toposes are closed under passage to presheaves.
3. Show that type-theoretic model toposes are closed under accessible left exact localizations.

We take the last two in reverse order.
Outline

1 Type-theoretic model toposes
2 Left exact localizations
3 Injective model structures
4 Remarks
Let $S$ be a set of morphisms in a type-theoretic model topos $E$.

**Definition**

A fibrant object $Z \in E$ is (internally) $S$-local if $Z^f : Z^B \to Z^A$ is an equivalence in $E$ for all $f : A \to B$ in $S$.

These are the fibrant objects of a **left Bousfield localization** model structure $\mathcal{L}_S E$ on the same underlying category $E$. It is **left exact** if fibrant replacement in $\mathcal{L}_S E$ preserves homotopy pullbacks in $E$.

**Example**

If $E = [C^{\text{op}}, \text{Set}]$ and $C$ is a site with covering sieves $R \to \mathfrak{X}c$, then $Z^R$ is the object of local/descent data. Thus the local objects are the sheaves/stacks.
Lemma

There is a loc. rep. notion of fibred structure whose $\mathbb{F}_S$-structured maps are the fibrations $X \to Y$ that are $S$-local in $\mathcal{E}/Y$.

Sketch of proof.

Define $\text{isLocal}_S(X)$ using the internal type theory, and let an $\mathbb{F}_S$-structure be an $\mathbb{F}$-structure and a section of $\text{isLocal}_S(X)$. □
Left exact localizations as type-theoretic model toposes

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>There is a loc. rep. notion of fibred structure whose ( F_S )-structured maps are the fibrations ( X \to Y ) that are ( S )-local in ( \mathcal{E}/Y ).</td>
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<td>Define ( \text{isLocal}_S(X) ) using the internal type theory, and let an ( F_S )-structure be an ( F )-structure and a section of ( \text{isLocal}_S(X) ).</td>
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<th>Theorem</th>
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<td>If ( S )-localization is left exact, ( \mathcal{L}_S \mathcal{E} ) is a type-theoretic model topos.</td>
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<td>Using Rijke–S.–Spitters and Anel–Biedermann–Finster–Joyal (forthcoming), if we close ( S ) under homotopy diagonals, the above ( F_S )-structured maps also coincide with the fibrations in ( \mathcal{L}_S \mathcal{E} ).</td>
</tr>
</tbody>
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Outline

1. Type-theoretic model toposes
2. Left exact localizations
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4. Remarks
$\mathcal{E}$ = a type-theoretic model topos, $\mathcal{D} = a$ small (enriched) category, $[\mathcal{D}^{op}, \mathcal{E}] = the$ presheaf category.

**Warning #1**

It’s essential that we allow presheaves over $(\infty, 1)$-categories (e.g. simplicially enriched categories) rather than just 1-categories. But for simplicity here, let’s assume $\mathcal{D}$ is unenriched.

**Warning #2**

In cubical cases, $[\mathcal{D}^{op}, \mathcal{E}]$ has an “intrinsic” cubical-type model structure, which (when $\mathcal{D}$ is unenriched) coincides with the ordinary cubical model constructed in the internal logic of $[\mathcal{D}^{op}, \text{Set}]$. However, this generally does not present the correct $(\infty, 1)$-presheaf category, as discussed by Thierry yesterday.
Theorem

The category $[\mathcal{D}^{\text{op}}, \mathcal{E}]$ of presheaves has an injective model structure such that:

1. The weak equivalences and cofibrations are pointwise.
2. It is right proper and Cisinski.
3. It presents the corresponding presheaf $(\infty, 1)$-category.

Thus it lacks only a suitable notion of fibred structure to be a type-theoretic model topos.
Injective model structures

Theorem

The category \([\mathcal{D}^{\text{op}}, \mathcal{E}]\) of presheaves has an injective model structure such that:

1. The weak equivalences and cofibrations are pointwise.
2. The fibrations are . . . ?????
3. It is right proper and Cisinski.
4. It presents the corresponding presheaf \((\infty, 1)\)-category.

Thus it lacks only a suitable notion of fibred structure to be a type-theoretic model topos.
Why pointwise isn’t enough

When is $X \in [\mathcal{D}^{\text{op}}, \mathcal{E}]$ injectively fibrant? We want to lift in

$$
\begin{array}{ccc}
A & \xrightarrow{g} & X \\
\downarrow{\sim} & & \downarrow{\sim} \\
B & & \\
\end{array}
$$

where $i : A \to B$ is a pointwise acyclic cofibration.
If $X$ is pointwise fibrant, then for all $d \in \mathcal{D}$ we have a lift

$$
\begin{array}{ccc}
A_d & \xrightarrow{g_d} & X_d. \\
\downarrow{\sim} & & \downarrow{h_d} \\
B_d & & \\
\end{array}
$$

These may not fit together into a natural transformation $B \to X$, but they do form a homotopy coherent natural transformation.
The coherent morphism coclassifier

**Lemma**

The notion of coherent natural transformation is representable. That is, there is a **coherent transformation coclassifier** $C^D(Y)$ (classically called the **cobar construction**) with a natural bijection

$$h : X \xrightarrow{\sim} Y \quad \leftarrow \quad h : X \rightarrow C^D(Y)$$

- The (strictly natural) identity $X \xrightarrow{\sim} X$ corresponds to a canonical map $\nu_X : X \rightarrow C^D(X)$.
- $\nu_X$ is always a **pointwise acyclic cofibration**!
Injective fibrancy

**Theorem (S.)**

\[ X \in \left[ \mathcal{D}^{\text{op}}, \mathcal{E} \right] \text{ is injectively fibrant if and only if it is pointwise fibrant and } \nu_X : X \to C^\mathcal{D}(X) \text{ has a retraction } r : C^\mathcal{D}(X) \to X. \]
Theorem (S.)

$X \in [\mathcal{D}^{\text{op}}, \mathcal{E}]$ is injectively fibrant if and only if it is pointwise fibrant and $\nu_X : X \to \mathcal{C}^\mathcal{D}(X)$ has a retraction $r : \mathcal{C}^\mathcal{D}(X) \to X$.

Proof of “only if”.

If $X \in [\mathcal{D}^{\text{op}}, \mathcal{E}]$ is injectively fibrant, then since $\nu_X$ is a pointwise acyclic cofibration we have a lift:
Injective fibrancy

**Theorem (S.)**

\(X \in [\mathcal{D}^{\text{op}}, \mathcal{E}]\) is injectively fibrant if and only if it is pointwise fibrant and \(\nu_X : X \to C^\mathcal{D}(X)\) has a retraction \(r : C^\mathcal{D}(X) \to X\).

**Proof of “if”**.

Given a pointwise acyclic cofibration \(i : A \to B\) and a map \(g : A \to X\), we construct a coherent \(h : B \rightleftharpoons X\) with \(h \circ i = g\).

\[
\begin{array}{c}
A \xrightarrow{g} X \\
i \downarrow \sim \\ B
\end{array}
\]

We have \(\overline{h} : B \to C^\mathcal{D}(X)\); define \(k = r \circ \overline{h} : B \to X\). Since \(h \circ i = g\) is strict, \(\overline{h} \circ i = \nu_X \circ g\), and \(k \circ i = r \circ \overline{h} \circ i = r \circ \nu_X \circ g = g\). \(\square\)
Given \( f : X \to Y \), define a factorization by pullback:

\[
\begin{array}{ccc}
X & \xrightarrow{\nu_X} & C^D(f) \\
\lambda_f & \downarrow & \downarrow \rho_f \\
Y & \xrightarrow{\nu_Y} & C^D(Y)
\end{array}
\]

**Theorem (S.)**

\( f : X \to Y \) is an injective fibration if and only if it is a pointwise fibration and \( \lambda_f \) has a retraction \( r : C^D(f) \to X \) over \( Y \).
A notion of injective fibration structure

Note $C^D$ is a fibred pointed endofunctor of $[D^{op}, E]$. Thus, if we define an $F^D$-structure to be a pointwise $F$-structure and a $C^D$-algebra structure, we get a locally representable notion of fibred structure for the injective fibrations in $[D^{op}, E]$.

**Theorem**

$[D^{op}, E]$ is a type-theoretic model topos with $F^D$.

This completes the main result.
Outline

1. Type-theoretic model toposes
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Future work

1. Are these universes closed under higher inductive types?
2. Do Grothendieck $(\infty, 1)$-toposes model *cubical* type theory? (Perhaps with cubically enriched type-theoretic model toposes?)
3. How much of this works in a constructive metatheory?
4. What about *elementary* $(\infty, 1)$-toposes? (E.g. by Yoneda?)