



# Univalence and completeness of Segal objects

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# Outline

Introduction

Univalence

Rezk completeness

Comparison of univalence and completeness

Univalent and Rezk completion

Outlook

## Type theoretic model categories

Definition (ad hoc)

A *type theoretic model category*  $\mathbb{M}$  is a model category such that its associated category  $\mathbb{C} := \mathbb{M}^f$  of fibrant objects is a type theoretic fibration category.

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The Quillen model structure  $(\mathbf{S}, \text{Kan})$ .

### Recall

1. Complete Segal spaces are Reedy fibrant simplicial objects in  $(\mathbf{S}, \text{Kan})$  satisfying *the Segal conditions* and *the completeness condition*.

## Type theoretic model categories

Definition (sort of)

A *type theoretic model category*  $\mathbb{M}$  is a model category such that its associated category  $\mathbb{C} := \mathbb{M}^f$  of fibrant objects is a type theoretic fibration category.

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Example

The Quillen model structure  $(\mathbf{S}, \text{Kan})$ .

Recall

2. There is a model structure  $(s\mathbf{S}, \text{CS})$  whose fibrant objects are the complete Segal spaces.

$\rightsquigarrow$  Classical model for  $(\infty, 1)$ -category theory.

Fix a type theoretic model category  $\mathbb{M}$  with associated category  $\mathbb{C}$  of fibrant objects.

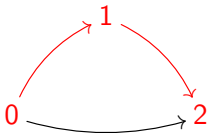


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Let  $\iota_n: I_n \hookrightarrow \Delta^n$  be the  $n$ -th *spine inclusion*.

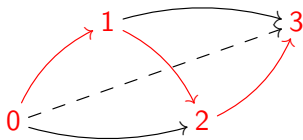
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### Definition

The  $n$ -th Segal map associated to a simplicial object  $X$  in  $\mathbb{M}$  is the map

$$\iota_n \setminus X: \Delta^n \setminus X \rightarrow I_n \setminus X.$$

Let  $X \in \mathbf{sC}$  be a simplicial object in  $\mathbb{C}$ .

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The  $n$ -th Segal map associated to a simplicial object  $X$  in  $\mathbb{M}$  is the map

$$\xi_n: X_n \rightarrow (X_{1/X_0})_S^n.$$

## Definition

Let  $X \in s\mathbb{C}$  be a simplicial object in  $\mathbb{C}$ .

1.  $X$  is *sufficiently fibrant* if both the 2-Segal map

$$\xi_2: X_2 \rightarrow X_1 \times_{X_0} X_1$$

and the boundary map

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2. Let  $X$  be sufficiently fibrant. We say that  $X$  is a *Segal object* (*strict Segal object*) if the associated Segal maps

$$\xi_n: X_n \rightarrow (X_1/X_0)_S^n$$

are homotopy equivalences (isomorphisms) in  $\mathbb{C}$ .

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for

$$\text{Linv}(x, y, f) := \sum_{g:X_1(y,x)} \sum_{\sigma:X_2(f,g)} d_1 \sigma =_{X_1(x,x)} s_0 x,$$

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Let  $p: E \twoheadrightarrow B$  be a fibration in  $\mathbb{C}$ .

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- ▶ There is a nerve construction

$$N: \text{ICat}(\mathbb{C}) \rightarrow s\mathbb{C}$$

whose image consists exactly of the objects in  $s\mathbb{C}$  whose Segal objects are isomorphisms.

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## Proposition

*Let  $p: E \twoheadrightarrow B$  be a fibration in  $\mathbb{C}$ . Then  $p$  is a univalent fibration in  $\mathbb{C}$  if and only if the Segal object  $N\text{Fun}(p)$  is univalent.*

# Rezk Completeness

Let  $X \in s\mathbb{C}$  be a simplicial object in  $\mathbb{C}$ . Recall the Kan extension

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### Definition

A Reedy fibrant Segal object  $X$  is *complete* if the functor

$$\_ \setminus X : (\mathbf{S}, \mathbf{QCat})^{op} \rightarrow \mathbb{M}$$

is a right Quillen functor.

A map  $\mathcal{C} \rightarrow \mathcal{D}$  between quasi-categories is a quasi-fibration if and only if it has the right lifting property against

1. all inner horn inclusions  $\{h_i^n: \Lambda_i^n \rightarrow \Delta^n\}$ , and
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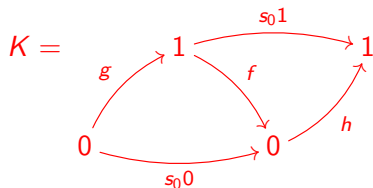
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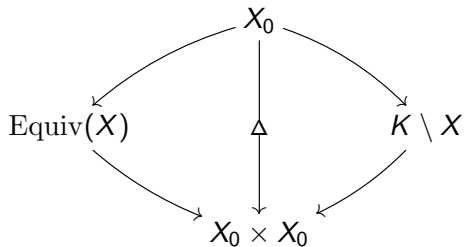
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A Reedy fibrant Segal object  $X$  is *complete* if the object  $K \setminus X$  is a path object for  $X_0$ .

# Comparison of univalence and completeness

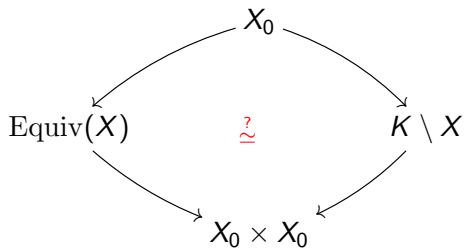
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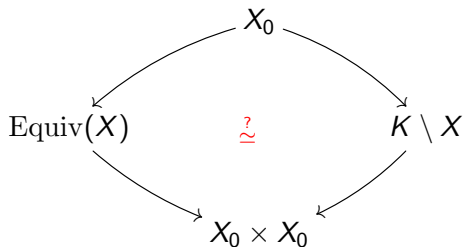
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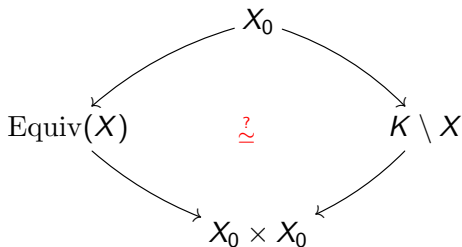


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### Theorem

*Let  $X$  be a Segal object in  $\mathbb{C}$ . Then  $X$  is univalent if and only if its Reedy fibrant replacement  $\mathbb{R}X$  is complete.*

## Comparison of univalence and completeness

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$$\begin{array}{ccc}
 & X_0 & \\
 \swarrow & & \searrow \\
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 \searrow & & \swarrow \\
 & X_0 \times X_0 &
 \end{array}$$

$$\text{Equiv}(X) \simeq \text{Equiv}(\mathbb{R}X) \simeq K \setminus \mathbb{R}X$$

### Corollary

*Let  $X$  be a fibration in  $\mathbb{C}$ . Then  $p$  is a univalent fibration if and only if the Segal object  $\mathbb{R}N(\text{Fun}(p))$  is complete.*

# Univalent and Rezk completion

## Univalent completion

Let  $\mathbb{M}$  be a type theoretic model category with an (h-epi,h-mono)-factorization (e.g.  $\mathbb{M}$  combinatorial).

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Let  $\pi: \tilde{U} \rightarrow U$  be a univalent fibration.

### Definition

We say that  $p: E \twoheadrightarrow B$  is *small* if it arises as the homotopy pullback of  $\pi$  along some map  $B \rightarrow U$ .



## Definition

Let  $p: E \twoheadrightarrow B$  be a small fibration in  $\mathbb{C}$ . We say that a homotopy cartesian square

$$\begin{array}{ccc}
 E & \longrightarrow & u(E) \\
 p \downarrow & & \downarrow u(p) \\
 B & \xrightarrow{\iota} & u(B)
 \end{array}$$

is a *univalent completion* of  $p$  if the fibration  $u(p) \in \mathbb{C}$  is small and univalent, and the map  $\iota: B \rightarrow u(B)$  is a  $(-1)$ -connected cofibration.

## Proposition

*For every fibration  $p: E \rightarrow B$  in  $\mathbb{C}$  there is a univalent completion*

$$\begin{array}{ccc} E & \longrightarrow & u(E) \\ p \downarrow & & \downarrow u(p) \\ B & \xrightarrow{\iota} & u(B). \end{array}$$

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$$\begin{array}{ccc}
 E & \longrightarrow & u(E) \\
 p \downarrow & & \downarrow u(p) \\
 B & \xrightarrow{u} & u(B).
 \end{array}$$

Proof.

$$\begin{array}{ccc}
 E & \longrightarrow & \tilde{U} \\
 p \downarrow & \lrcorner & \downarrow \pi \\
 B & \xrightarrow{b} & U
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Proof.

$$\begin{array}{ccccc}
 E & \longrightarrow & (b_{-1})^* \tilde{U} & \longrightarrow & \tilde{U} \\
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# Rezk completion

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Given a map  $f : X \rightarrow Y$  of Segal objects in  $\mathbb{C}$ , consider

$$\begin{array}{ccc}
 \text{Eq}f & \longrightarrow & \text{Equiv } Y \\
 \downarrow & \lrcorner & \downarrow \\
 f \downarrow Y & \longrightarrow & Y_1 \\
 \downarrow & \lrcorner & \downarrow (s,t) \\
 X_0 \times Y_0 & \xrightarrow{f_0 \times 1} & Y_0 \times Y_0
 \end{array}$$

## Definition

Let  $f: X \rightarrow Y$  be a map between Segal objects in  $\mathbb{M}$ . We say that

1.  $f$  is *fully faithful* if the natural map  $X_1 \rightarrow (f_0 \times f_0)^* Y_1$  over  $X_0 \times X_0$  is a weak equivalence.
2.  $f$  is *essentially surjective* if the fibration  $(\text{Eq}f)_{-1} \rightarrow Y_0$  is acyclic.
3.  $f$  is a *DK-equivalence* if it is fully faithful and essentially surjective.



## Theorem

For every fibration  $p: E \rightarrow B$ , the univalent completion

$$\begin{array}{ccc}
 E & \longrightarrow & u(E) \\
 p \downarrow & & \downarrow u(p) \\
 B & \xrightarrow{\iota} & u(B)
 \end{array}$$

induces a DK-equivalence

$$\mathbb{R}N(\iota): \mathbb{R}N(p) \rightarrow \mathbb{R}N(u(p))$$

from the Segal object  $\mathbb{R}N(p)$  to the complete Segal object  $\mathbb{R}N(u(p))$  in  $\mathbb{C}$ .

# Outlook





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- ▶ Deduce existence of Rezk completion from univalent completion and vice versa, depending on the context.

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- ▶ Deduce existence of Rezk completion from univalent completion and vice versa, depending on the context.
- ▶ Coming from simplicial homotopy theory, this suggests that univalent fibrations might be the fibrant objects in some fibration category?

# Thank you!

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