On Church's Thesis in Cubical Assemblies

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Given $e : \mathbb{N}$, write $\varphi_e : \mathbb{N} \to \mathbb{N}$ for the partial function computed by the eth Turing machine.

Definition

Church's thesis is the statement that for every function $f : \mathbb{N} \to \mathbb{N}$ there merely exists $e : \mathbb{N}$ such that $\varphi_e = f$.

Theorem (Kleene '52)

Church's thesis is consistent with Heyting arithmetic.

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Church's thesis holds in the internal language of the effective topos.

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Theorem

Assume function extensionality. Then the "untruncated version" of Church's thesis below is false:

$$\prod_{f:\mathbb{N}\to\mathbb{N}} \sum_{e:\mathbb{N}} \prod_{n:\mathbb{N}} \varphi_e(n) = f(n)$$

Proof.

This says there is a function $F \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ such that for all $f \colon \mathbb{N} \to \mathbb{N}$, $f = \varphi_{F(f)}$. Note that any such function is injective, and so $\mathbb{N}^{\mathbb{N}}$ has decidable equality. This implies there is a function $\mathbb{N} \to 2$ that decides whether or not each Turing machine halts, which is non computable. $\square$

What about realizability models of univalent type theory? We work with internal cubical sets in the category of assemblies: cubical assemblies. (Cubical assemblies have also been investigated by Awodey and Frey, and simplicial assemblies by Stekelenburg.)

Theorem (Swan, Uemura)

1. *Church's thesis does not hold in cubical assemblies.*
2. *Church's thesis holds in a reflective subuniverse of cubical assemblies.*

Cubical assemblies are a regular locally cartesian closed category, so they can be viewed as a model of extensional type theory with propositional truncation (Awodey-Bauer).

In the interpretation of extensional type theory in a locally cartesian closed category:

- Types in context Γ are interpreted as maps $A \to \Gamma$.
- Terms are interpreted as sections $\Gamma \to A$ (we will also refer to sections as *points*).
- Two terms are propositionally equal only if they are equal.
- Hence if a type is an hproposition it has at most one section.
- Propositional truncation "strictly identifies points."

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Orton-Pitts models of cubical type theory consist of two levels:

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- Terms are interpreted as morphisms (as in the extensional level).
- Two terms $\sigma, \tau \colon \Gamma \to A$ are propositionally equal if they are homotopic: i.e. there is a map $h \colon \mathbb{I} \times \Gamma \to A$ such that $h \circ \delta_0 = \sigma$ and $h \circ \delta_1 = \tau$ (constant over Γ).
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NB: By Kraus' paradox the truncation map $\sum_{e:\mathbb{N}} \prod_{n:\mathbb{N}} \varphi_e(n) = f(n) \longrightarrow \| \sum_{e:\mathbb{N}} \prod_{n:\mathbb{N}} \varphi_e(n) = f(n) \|$ is a monomorphism in *any* model of univalent type theory.

Theorem (Uemura)

Cubical assemblies form a model of cubical type theory. In particular the method of Licata-Orton-Pitts-Spitters can be used to construct a univalent universe.

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Theorem (Swan, Uemura)

The Coquand-Huber-Mörtberg construction of propositional truncation and suspensions can be carried out in any Orton-Pitts category with cofibrant non-dependent W-types with reductions. The same technique works for localization/nullification.

Theorem (Swan)

Cofibrant W-types with reductions exist in any category of presheaf assemblies as long as cofibrations are locally decidable.

Theorem (Swan, Uemura)

(In the internal language of a regular locally cartesian closed category) Church's thesis does not hold in cubical sets, even if it holds in the metatheory.

Main ideas in proof:

- The constant presheaf functor from sets to cubical sets has a left adjoint given by global sections, and also a right adjoint. Objects in the image of the right adjoint are sometimes called *codiscrete*.
- The right adjoint maps to hpropositions.
- One can use the above to construct a function from global sections of $\| \sum_{e:\mathbb{N}} \prod_{n:\mathbb{N}} \varphi_e(n) = f(n) \|$ to global sections of $\sum_{e:\mathbb{N}} \prod_{n:\mathbb{N}} \varphi_e(n) = f(n)$ for each function $f : \mathbb{N} \to \mathbb{N}$.
- We recall untruncated Church's thesis is false.

We emphasise that the features of cubical sets that we use in the proof are "good" features that we want models of type theory to have.

- They are a regular locally cartesian closed category, allowing us to apply the Orton-Pitts method (but also allow us to prove untruncated Church's thesis is false).
- Dependent products and sums are the same in the extensional level and the cubical level, but with some extra structure.
- Propositional truncation adds new paths but does not add (essentially) any new points.

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We focus on the property that propositional truncation does not add (essentially) any new points.

Idea: We construct a model where $\|A\|$ is forced to contain more points. In particular whenever we are given any type A, a function f, and a function $\| \sum_{e:\mathbb{N}} f = \varphi_e \| \to A$ we add a new point to $\|A\|$. We do this without adding any new elements to $\mathbb{N}$, $\mathbb{N} \to \mathbb{N}$ or to $\perp$.

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We do this using a reflective subuniverse of *null types*.

Definition (Rijke, Shulman, Spitters)

Suppose we are given a family of types $B\colon A \to \mathcal{U}$. We say a type X is *B-null* if for every $a : A$, the constant function map $X \to X^{B(a)}$ is an equivalence.

The *nullification* of a type X is a type $\mathcal{L}_B(X)$ and a map $\alpha_X^B\colon X \to \mathcal{L}_B(X)$ such that for every B-null type Y, every map $X \to Y$ factors uniquely through α_X^B.

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Theorem (Rijke, Shulman, Spitters)

Suppose that $B\colon A \to \mathcal{U}$ is a family of hpropositions. B-null types form a reflective subuniverse. They are closed under dependent products, sums and identity types and contain the unit type. The universe of B-null types is itself B-null (and univalent).

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Note that the general theorem includes "limits but not colimits." E.g. If $B(a)$ is empty for some $a : A$, then the only B-null type is the unit type $\top$. We give a criterion for when B-null are non-trivial.

Proposition

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Suppose that $B\colon A \to \mathcal{U}$ is a family of hpropositions. Propositional truncation for B-null types can be constructed as the nullification of the usual propositional truncation, i.e. $\mathcal{L}_B(\|X\|)$.

- Propositional truncation and nullification both exist in cubical assemblies.
- Nullification adds new global sections from which we can't uniformly compute global sections of the untruncated type.

Theorem (Swan, Uemura)

Suppose that we are given,

1. *A family of types* $C : A \to \mathcal{U}$.
2. *Both* A *and* C *are constructed using only* $\mathbb{N}$*, dependent products and coproducts and identity types.*
3. *The statement "every* $C(a)$ *is merely inhabited" holds in the extensional level of* $\mathcal{E}$

Define $B(a) := \|C(a)\|$*. Then,*

1. *Every constant cubical set is* B*-null.*
2. B*-null types are closed under binary coproduct.*
3. *The interpretation of* $\prod_{a:A} \|C(a)\|$ *in* B*-null types is equal to* $\prod_{a:A} \mathcal{L}_B \|C(a)\|$*, and is inhabited.*

Corollary

Church's thesis holds in the reflective subuniverse of B*-null sets in cubical assemblies, where* $A := \mathbb{N} \to \mathbb{N}$ *and* $B(f) := \sum_{e:\mathbb{N}} \varphi_e = f$.

For more details see *On Church's Thesis in Cubical Assemblies* arXiv:1905.03014.

Remaining open problems:

1. What is the consistency strength of univalent type theory with Church's thesis? (Our proof gives an upper bound of "type theory with W-types" which is much higher than "type theory without W-types")
2. What is the right way to define the effective ∞-topos?
3. Is there a model of univalent type theory where Church's thesis and countable choice hold?
4. What about propositional resizing? (Uemura: propositional resizing does not hold in cubical assemblies).
5. What about extended Church's thesis (every partial function with an almost negative domain is extended by a computable function)?

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Thank you for your attention!