Internal Languages of Higher Categories

Karol Szumiło

University of Leeds

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What is a counterpart for HoTT?

Joint work with Chris Kapulkin:

Theorem (2017)

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Theorem (in progress)

The homotopy theory of comprehension categories with Id, Π *and* Σ *satisfying functional extensionality is equivalent to the homotopy theory of locally cartesian closed quasicategories.*

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A functor $F: \mathcal{C} \to \mathcal{D}$ is homotopical if it preserves weak equivalences.

It is a DK-equivalence if $\operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ is essentially surjective and $L^H\mathcal{C}(X, Y) \to L^H\mathcal{D}(FX, FY)$ are weak homotopy equivalences, i.e., it induces an equivalences of $(\infty, 1)$-categories.

$$\mathsf{CompCat}_{\mathsf{Id},\Pi,\Sigma} \longrightarrow \mathsf{LCCQ}$$

comprehension categories

locally cartesian closed quasicategories

$$\mathsf{CompCat}_{\mathsf{Id},\Pi,\Sigma} \longrightarrow \Pi\text{-}\mathsf{Trb} \longrightarrow \mathsf{LCCQ}$$

comprehension categories — Π-tribes — locally cartesian closed quasicategories

$$
\begin{tikzcd}
 & \text{semisimplicial } \Pi\text{-tribes} & \\
 & \Pi\text{-sTrb} \arrow[d] \arrow[dr] & \\
\mathrm{CompCat}_{\mathrm{Id},\Pi,\Sigma} \arrow[r] & \Pi\text{-Trb} \arrow[r] & \mathrm{LCCQ} \\
\text{comprehension categories} & \Pi\text{-tribes} & \begin{array}{c}\text{locally cartesian}\\ \text{closed quasicategories}\end{array}
\end{tikzcd}
$$

semisimplicial Π-tribes

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 & \Pi\text{-sTrb} \arrow[d] \arrow[dr] & \\
\mathsf{CompCat}_{\mathsf{Id},\Pi,\Sigma} \arrow[r] & \Pi\text{-Trb} \arrow[r] \arrow[u, dashed, bend left] \arrow[l, dashed, bend right] & \mathsf{LCCQ}
\end{tikzcd}$$

comprehension categories | Π-tribes | locally cartesian closed quasicategories

A tribe is a category $\mathcal{C}$ with a subcategory of fibrations.

- T1 There is a terminal object and all objects are fibrant.
- T2 There are pullbacks along fibrations and fibrations are stable under pullback.
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$$
\begin{tikzcd}
a \arrow[d, two heads] \\
b
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & Pa \arrow[dr, two heads] & \\
a \arrow[ur, tail, "\sim"] \arrow[rr] & & a \times_b a
\end{tikzcd}
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\qquad
\begin{tikzcd}
a \arrow[r] \arrow[d, tail, "\sim"'] & c \arrow[d, two heads] \\
Pa \arrow[r] \arrow[ur, dashed] & Pa
\end{tikzcd}
$$

A path object on a is given by a factorization $a \rightarrowtail\!\!\!\!\!\sim\ Pa \twoheadrightarrow a \times a$.

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A homomorphism of tribes is a functor that preserves fibrations, anodyne morphisms, terminal object and pullbacks along fibrations.

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A Π-tribe is a tribe $\mathcal{T}$ such that for all fibrations $p\colon a \twoheadrightarrow b$, the pullback functor $p^*\colon T \downarrow b \to T \downarrow a$ has a right adjoint Π_p that is a homomorphism.

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It is a weak equivalence if it induces an equivalence on homotopy categories.

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A semisimplicial Π-tribe is a Π-tribe that is enriched in semisimplicial sets such that

- cotensors by finite semisimplicial sets exist and the "pullback cotensor property" is satisfied;
- cotensors preserve anodyne morphisms;
- adjunctions $p^* \vdash \Pi_p$ are semisimplicial.

A frame in a Π-tribe $\mathcal{T}$ is a Reedy fibrant homotopically constant semisimplicial object in $\mathcal{T}$.

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$$\begin{tikzcd} \Pi\text{-sTrb} \arrow[loop left, "\widehat{\mathrm{Fr}}"] \arrow[r, "U"'] & \Pi\text{-Trb} \arrow[l, bend right, "\mathrm{Fr}"'] \end{tikzcd}$$

$$\mathrm{Fr}\,U\mathcal{T}' \xleftarrow{\sim} \widehat{\mathrm{Fr}}\mathcal{T}' \xrightarrow{\sim} \mathcal{T}' \qquad\qquad U\,\mathrm{Fr}\,\mathcal{T} \xrightarrow{\sim} \mathcal{T}$$

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$\mathcal{C}$ has finite limits if for all finite simplicial sets K and $x\colon K \to \mathcal{C}$, the functor $\mathcal{C}^K(\mathrm{const}\,-, x)$ is representable.

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A morphism of LCCQs is a functor (simplicial map) that preserves finite limits and local exponentials.

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A morphism of LCCQs is a functor (simplicial map) that preserves finite limits and local exponentials.

It is an categorical equivalence if it has an inverse up to natural equivalence.

$$\begin{tikzcd}
\Pi\text{-sTrb} \arrow[d, "U"] \arrow[dr] \\
\Pi\text{-Trb} \arrow[u, bend left, "\mathsf{Fr}"] \arrow[r] & \mathsf{LCCQ}
\end{tikzcd}$$

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Outline of the proof:

- Show that Π-sTrb and LCCQ are fibration categories.

$$\begin{tikzcd}
\Pi\text{-}\mathsf{sTrb} \arrow[d, "U"] \arrow[dr, "\mathsf{N}_f"] & \\
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Outline of the proof:

- Show that Π-sTrb and LCCQ are fibration categories.
- Implement the rightward functors with a convenient construction: the quasicategory of frames.

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- Show that Π-sTrb and LCCQ are fibration categories.
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- Show that N_f is an exact functor with Approximation Properties.

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- Show that Π-sTrb and LCCQ are fibration categories.
- Implement the rightward functors with a convenient construction: the quasicategory of frames.
- Show that N_f is an exact functor with Approximation Properties.
- For that use tribes of representable injectively fibrant presheaves.

A fibration category is a category with subcategories of weak equivalences and fibrations.

- F1 There is a terminal object and all objects are fibrant.
- F2 Pullbacks along fibrations exist and (acyclic) fibrations are stable under pullback.
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An exact functor between fibration categories is a functor that preserves weak equivalences, fibrations, terminal object and pullbacks along fibrations.

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Every tribe is a fibration category and every homomorphism is exact.

LCCQ is a fibration category (fibrations are inner isofibrations).

A fibration between tribes $F\colon \mathcal{S} \twoheadrightarrow \mathcal{T}$:

$$
\begin{tikzcd}
\mathcal{S} \arrow[d, "F"'] \\
\mathcal{T}
\end{tikzcd}
$$

A fibration between tribes $F\colon \mathcal{S} \twoheadrightarrow \mathcal{T}$:

$$\begin{tikzcd}
\mathcal{S} \arrow[dd, "F"'] & a \arrow[r, "\cong"] & b \\
& & \\
\mathcal{T} & Fa \arrow[r, "\cong"] & y
\end{tikzcd}$$

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$$
\begin{tikzcd}
\mathcal{S} \arrow[dd, "F"'] & a \arrow[r, "\cong"] & b & a \arrow[rr] \arrow[dr, tail, "\sim"'] & & c \\
& & & & b \arrow[ur, two heads] & \\
\mathcal{T} & Fa \arrow[r, "\cong"] & y & Fa \arrow[rr] \arrow[dr, tail, "\sim"'] & & Fc \\
& & & & y \arrow[ur, two heads] &
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\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r] \arrow[d, tail, "\sim"'] & c \arrow[d, two heads] \\
b \arrow[r] \arrow[ur] & d \\
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$$

The category Π-sTrb is a fibration category.

A path object on $\mathcal{T} \in$ Π-sTrb:

$$
\mathcal{T} \xrightarrow{\sim} \mathcal{T}_R^Z \twoheadrightarrow \mathcal{T} \times \mathcal{T}
$$

where $Z = \{\bullet \xleftarrow{\sim} \bullet \xrightarrow{\sim} \bullet\}$.

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The "initial projection" functor $\mathsf{N}\,\mathsf{I}\mathcal{C} \to \mathcal{C}$ creates weak equivalences in $\mathsf{I}\mathcal{C}$.

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$\mathsf{I}\mathcal{C}$ is a homotopical inverse category and $\mathsf{N}\,\mathsf{I}\mathcal{C} \to \mathcal{C}$ is a localization.

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If $\mathcal{T}$ is a Π-tribe, define a simplicial set $\mathsf{N}_f\,\mathcal{T}$ (quasicategory of frames):

$$(\mathsf{N}_f)_m = \{x : \mathsf{I}[m] \to \mathcal{T} \mid x \text{ is homotopical and Reedy fibrant}\}.$$

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$\mathrm{I}\mathcal{C}$ is a homotopical inverse category and $\mathrm{N}\,\mathrm{I}\mathcal{C} \to \mathcal{C}$ is a localization.

If $\mathcal{T}$ is a Π-tribe, define a simplicial set $\mathrm{N}_f\,\mathcal{T}$ (quasicategory of frames):

$$(\mathrm{N}_f)_m = \{x \colon \mathrm{I}[m] \to \mathcal{T} \mid x \text{ is homotopical and Reedy fibrant}\}.$$

$\mathrm{N}_f\,\mathcal{T}$ is a locally cartesian closed quasicategory and $\mathrm{N}_f \colon \Pi\text{-sTrb} \to \mathrm{LCCQ}$ is an exact functor.

Theorem (Cisinski)

If $F: \mathcal{C} \to \mathcal{D}$ is an exact functor between fibration categories, then the following are equivalent.

- *F is a DK-equivalence.*
- *F is a weak equivalence.*
- *F satisfies the Approximation Properties:*

App1 *it reflects weak equivalences,*

App2

$$\begin{tikzcd} x \arrow[r] & Fb & & b \\ x' \arrow[u, "\sim"] \arrow[r, "\sim"'] & Fa \arrow[u] & & a. \arrow[u] \end{tikzcd}$$

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We need to show that N_f satisfies App2.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathsf{N}_f\,\mathcal{T} & & \mathcal{T} \\
\mathcal{C}' \arrow[u, "\sim"] \arrow[r, "\sim"'] & \mathsf{N}_f\,\mathcal{S} \arrow[u] & & \mathcal{S} \arrow[u]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathrm{N}_f\,\mathcal{T} & \mathcal{T} \\
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$$

A simplicial presheaf A over a simplicial category $\mathcal{A}$ is representable if there is $\mathcal{A}(-, a) \xrightarrow{\sim} A$.

$$
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Consider the hammock localization $L^H \mathrm{I}\mathcal{C}$. Then the category $R\mathcal{C}$ of representable injectively fibrant simplicial presheaves over $L^H \mathrm{I}\mathcal{C}$ is a Π-tribe (with injective fibrations). Moreover, $\mathsf{N}_f\, R\mathcal{C}$ is equivalent to $\mathcal{C}$.

An object of $\mathcal{S}$ consists of

- presheaves X, $\widetilde{X}$ over $L^H \mathrm{I}\mathcal{C}$ with X injectively fibrant,
- an object $a \in \mathcal{T}$,
- an injective fibration $\widetilde{X} \twoheadrightarrow X \times L^H \mathcal{T}(F -, a)$

such that

- $\widetilde{X} \to X$ is a weak equivalence,
- there is a representation $L^H \mathrm{I}\mathcal{C}(-, x) \xrightarrow{\sim} \widetilde{X}$ that induces a weak equivalence $Fx \to a$.

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$\mathcal{S}$ is a Π-tribe.

An m-simplex of $\mathcal{C}'$ consists of

- an m-simplex $(X, \widetilde{X}, a)$ of $\mathrm{N}_f\, \mathcal{S}$,
- an m-simplex x of $\mathcal{C}$,
- a natural choice of representations $L^H \textstyle\int \mathcal{C}(-, x\varphi) \xrightarrow{\sim} \widetilde{X}_\varphi$ for $\varphi\colon [k] \hookrightarrow [m]$ inducing weak equivalences $Fx\varphi \to a\varphi$.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathsf{N}_f\,\mathcal{T} & & \mathcal{T} \\
\mathcal{C}' \arrow[u] \arrow[r] & \mathsf{N}_f\,\mathcal{S} \arrow[u] & & \mathcal{S} \arrow[u]
\end{tikzcd}
$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathsf{N}_f\,\mathcal{T} & & \mathcal{T} \\ \mathcal{C}' \arrow[u, "\sim"] \arrow[r] & \mathsf{N}_f\,\mathcal{S} \arrow[u] & & \mathcal{S} \arrow[u] \end{tikzcd}$$

$\mathcal{C}' \to \mathcal{C}$ is an acyclic fibration. In particular, $\mathcal{C}'$ is a LCCQ.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathrm{N}_f\,\mathcal{T} & & \mathcal{T} \\
\mathcal{C}' \arrow[u, "\sim"] \arrow[r, "\sim"'] & \mathrm{N}_f\,\mathcal{S} \arrow[u] & & \mathcal{S} \arrow[u]
\end{tikzcd}
$$

$\mathcal{C}' \to \mathcal{C}$ is an acyclic fibration. In particular, $\mathcal{C}'$ is a LCCQ.

$\mathcal{C}' \to \mathrm{N}_f\,\mathcal{S}$ preserves finite limits and satisfies quasicategorical Approximation Properties.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathsf{N}_f\,\mathcal{T} \\
\mathcal{C}' \arrow[u, "\sim"] \arrow[r, "\sim"'] & \mathsf{N}_f\,\mathcal{S} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T} & \widehat{\mathsf{Fr}}\mathcal{T} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathsf{Fr}\,\mathcal{T} \\
\mathcal{S} \arrow[u] & \widehat{\mathsf{Fr}}\mathcal{S} \arrow[l, "\sim"] \arrow[r, "\sim"'] \arrow[u] & \mathsf{Fr}\,\mathcal{S} \arrow[u]
\end{tikzcd}
$$

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