A General Framework for the Semantics of Type Theory

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CwF-semantics of Type Theory

Semantics of type theories based on *categories with families* (CwF) (Dybjer 1996).

- Martin-Löf type theory
- Homotopy type theory
- Homotopy type system (Voevodsky 2013) and two-level type theory (Annenkov, Capriotti, and Kraus 2017)
- Cubical type theory (Cohen et al. 2018)

Goal

To define a general notion of a "type theory" to unify the CwF-semantics of various type theories.

Outline

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Natural Models

An alternative definition of CwF.

Definition (Awodey 2018)

A *natural model* consists of...

- a category $\mathcal{S}$ (with a terminal object);
- a map $p : E \to U$ of presheaves over $\mathcal{S}$

such that p is representable: for any object $\Gamma \in \mathcal{S}$ and element $A \in U(\Gamma)$, the presheaf A^*E defined by the pullback

$$\begin{array}{ccc} A^*E & \longrightarrow & E \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\ よ\Gamma & \xrightarrow[A]{} & U \end{array}$$

is representable, where よ is the Yoneda embedding.

Interpreting Type Theory

Natural model	Type theory
$\Gamma \in \mathcal{S}$	$\Gamma \vdash \texttt{ctx}$
$A \in U(\Gamma)$	$\Gamma \vdash A\ \texttt{type}$
$a \in \{x \in E(\Gamma) \mid p(x) = A\}$	$\Gamma \vdash a : A$
$f : \Delta \to \Gamma$	context morphism
$A \cdot f \in U(\Delta)$	substitution
$a \cdot f \in E(\Delta)$	substitution

Representable Maps

The representable map $p : E \to U$ models *context comprehension*:

$$\begin{array}{ccc} \text{よ}\{A\} & \xrightarrow{\delta_A} & E \\ {\scriptstyle \pi_A}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \text{よ}\Gamma & \xrightarrow[A]{} & U \end{array} \qquad \text{よ}\{A\} \cong A^* E$$

Natural model	Type theory
$A : \text{よ}\Gamma \to U$	$\Gamma \vdash A\ \mathtt{type}$
$\{A\} \in \mathcal{S}$	$\Gamma, x : A \vdash \mathtt{ctx}$
$\pi_A : \{A\} \to \Gamma$	$(\Gamma, x : A) \to \Gamma$
$A \cdot \pi_A : \text{よ}\{A\} \to U$	$\Gamma, x : A \vdash A\ \mathtt{type}$
$\delta_A : \text{よ}\{A\} \to E$	$\Gamma, x : A \vdash x : A$

Variable Binding

Variable binding is modeled by the pushforward $p_* : [\mathcal{S}^{\mathsf{op}}, \mathbf{Set}]/E \to [\mathcal{S}^{\mathsf{op}}, \mathbf{Set}]/U$, that is, the right adjoint to the pullback p^*.

Example

$p_*(E \times U)$ is the presheaf of *type families*: for $\Gamma \in \mathcal{S}$ and $A : \text{よ}\Gamma \to U$, we have

$$\left\{ \begin{array}{ccc} & & p_*(E \times U) \\ & \nearrow & \downarrow \\ \text{よ}\Gamma & \xrightarrow[A]{} & U \end{array} \right\} \cong \left\{ \text{よ}\{A\} \dashrightarrow U \right\},$$

so a section of $p_*(E \times U)$ over A is a type family $\Gamma, x : A \vdash B\ \mathtt{type}$.

Modeling Type Constructors

Consider dependent function types (Π-types).

$$\frac{\Gamma \vdash A \ \texttt{type} \qquad \Gamma, x : A \vdash B \ \texttt{type}}{\Gamma \vdash \prod_{x:A} B \ \texttt{type}}$$

It is modeled by an operation Π such that

- $\Pi_\Gamma(A, B) \in U(\Gamma)$ for $\Gamma \in \mathcal{S}$, $A \in U(\Gamma)$ and $B \in U(\{A\})$;
- Π commutes with substitution.

Thus Π is a map $p_*(E \times U) \to U$ of presheaves.

Cubical Type Theory

To model (cartesian) cubical type theory, we need more representable maps.

Example

Contexts can be extended by an *interval*:

$$\frac{\Gamma \vdash \mathtt{ctx}}{\Gamma, \mathtt{i} : \mathbb{I} \vdash \mathtt{ctx}}$$

This is modeled by a presheaf $\mathbb{I}$ such that the map $\mathbb{I} \to 1$ is representable.

Summary on Natural Models

An (extended) natural model consists of...

- a category $\mathcal{S}$ (with a terminal object);
- some presheaves $U, E, \ldots$ over $\mathcal{S}$;
- some representable maps $p : E \to U, \ldots$;
- some maps $X \to Y$ of presheaves over $\mathcal{S}$ where X and Y are built up from $U, E, \ldots, p, \ldots$ using finite limits and pushforwards along the representable maps $p, \ldots$.

Outline

Representable Map Categories

Definition

A *representable map category* is a category $\mathcal{A}$ equipped with a class of arrows called representable arrows satisfying the following:

- $\mathcal{A}$ has finite limits;
- identity arrows are representable and representable arrows are closed under composition;
- representable arrows are stable under pullbacks;
- representable arrows are *exponentiable*: the pushforward $f_* : \mathcal{A}/X \to \mathcal{A}/Y$ along a representable arrow $f : X \to Y$ exists.

Example

$[\mathcal{S}^{\mathsf{op}}, \mathbf{Set}]$ with representable maps of presheaves.

Representable Map Categories

Proposition (Weber 2015)

Exponentiable arrows are stable under pullbacks.

Example

A category $\mathcal{A}$ with finite limits has structures of a representable map category:

Smallest one only isomorphisms are representable;

Largest one all exponentiable arrows are representable.

Also, given a class $\mathcal{R}$ of exponentiable arrows, we have the smallest structure of a representable map category containing $\mathcal{R}$.

Type Theories

Definition

A *type theory* is a (small) representable map category $\mathbb{T}$.

Definition

A *model* of a type theory $\mathbb{T}$ consists of...

- a category $\mathcal{S}$ with a terminal object;
- a morphism $(-)^{\mathcal{S}} : \mathbb{T} \to [\mathcal{S}^{\mathsf{op}}, \mathbf{Set}]$ of representable map categories, i.e. a functor preserving everything.

Cf. Functorial semantics of algebraic theories (Lawvere 1963), first-order categorical logic (Makkai and Reyes 1977)

Generalised Algebraic Theories

We give an example $\mathbb{G}$ of a type theory whose models are precisely the natural models.

Definition

We denote by $\mathbb{G}$ the opposite of the category of finitely presentable *generalised algebraic theories* (GATs) (Cartmell 1978).

From the general theory of locally presentable categories (Adámek and Rosický 1994), we get:

Proposition

$\mathbb{G}$ is essentially small and has finite limits, and $\mathbf{Fun}_{\mathsf{finlim}}(\mathbb{G}, \mathbf{Set})$ is equivalent to the category of generalised algebraic theories.

An Exponentiable Map of GATs

Definition

- $\mathfrak{U}_0 \in \mathbb{G}$ is the GAT consisting of a type constant A_0.
- $\mathfrak{E}_0 \in \mathbb{G}$ is the GAT consisting of a type constant A_0 and a term constant $\mathfrak{a}_0 : A_0$.
- $\partial_0 : \mathfrak{E}_0 \to \mathfrak{U}_0$ is the arrow in $\mathbb{G}$ represented by the inclusion $\mathfrak{U}_0 \to \mathfrak{E}_0$.

Proposition

$\partial_0 : \mathfrak{E}_0 \to \mathfrak{U}_0$ *in* $\mathbb{G}$ *is exponentiable.*

So $\mathbb{G}$ has the smallest structure of a representable map category containing ∂_0.

An Exponentiable Map of GATs

Example

Let Σ denote the finite GAT

$$\vdash B \ \texttt{type}$$
$$x_1 : B, x_2 : B \vdash C(x_1, x_2) \ \texttt{type}$$
$$x : B \vdash c(x) : C(x, x).$$

Then $(\partial_0)_*(E_0 \times \Sigma)$ is the finite GAT

$$\vdash A_0 \ \texttt{type}$$
$$x_0 : A_0 \vdash B(x_0) \ \texttt{type}$$
$$x_0 : A_0, x_1 : B(x_0), x_2 : B(x_0) \vdash C(x_0, x_1, x_2) \ \texttt{type}$$
$$x_0 : A_0, x : B(x_0) \vdash c(x_0, x) : C(x_0, x, x).$$

Representable Map Category of Finite GATs

Theorem

$\mathbb{G}$ is "freely generated by ∂_0" as a representable map category: for a representable map category $\mathcal{A}$ and a representable arrow $f : X \to Y$ in $\mathcal{A}$, there exists a unique, up to isomorphism, morphism $F : \mathbb{G} \to \mathcal{A}$ of representable map categories equipped with an isomorphism $F\partial_0 \cong f$.

Corollary

Models of $\mathbb{G} \simeq$ Natural models ($\simeq$ CwFs)

Outline

Bi-initial Models

Let $\mathbb{T}$ be a type theory.

Theorem

The 2-category $\mathbf{Mod}_{\mathbb{T}}$ of models of $\mathbb{T}$ has a bi-initial object.

Theory-model Correspondence

Definition

A $\mathbb{T}$-*theory* is a functor $\mathbb{T} \to \mathbf{Set}$ preserving finite limits. Put $\mathbf{Th}_{\mathbb{T}} := \mathbf{Fun}_{\mathsf{finlim}}(\mathbb{T}, \mathbf{Set})$.

Example

A $\mathbb{G}$-theory is a generalised algebraic theory.

Theory-model Correspondence

Definition

We define the *internal language* 2-functor $\mathsf{L}_{\mathbb{T}} : \mathbf{Mod}_{\mathbb{T}} \to \mathbf{Th}_{\mathbb{T}}$ as

$$\mathsf{L}_{\mathbb{T}}(\mathcal{S}) = \left(\mathbb{T} \xrightarrow{(-)^{\mathcal{S}}} [\mathcal{S}^{\mathsf{op}}, \mathbf{Set}] \xrightarrow{\mathsf{X} \mapsto \mathsf{X}(1)} \mathbf{Set} \right).$$

Theorem

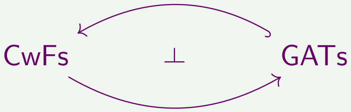
$\mathsf{L}_{\mathbb{T}}$ *has a left bi-adjoint with invertible unit.*

$$\mathbf{Mod}_{\mathbb{T}} \underset{\mathsf{L}_{\mathbb{T}}}{\overset{\mathsf{M}_{\mathbb{T}}}{\underset{\perp}{\rightleftarrows}}} \mathbf{Th}_{\mathbb{T}}.$$

Theory-model Correspondence

Example

When $\mathbb{T} = \mathbb{G}$, we get a bi-adjunction



Conclusion

Definition

A *type theory* is a (small) representable map category $\mathbb{T}$.

Further results and future directions:

- Logical framework for representable map categories
- Application: canonicity by gluing representable map categories (instead of gluing models)?
- What can we say about the 2-categoty $\mathbf{Mod}_{\mathbb{T}}$?
- What can we say about the category $\mathbf{Th}_{\mathbb{T}}$?
- Variations: internal type theories? $(\infty, 1)$-type theories?

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The Bi-initial Model

For a type theory $\mathbb{T}$, we define a model $\mathcal{I}(\mathbb{T})$ of $\mathbb{T}$:

- the base category is the full subcategory of $\mathbb{T}$ consisting of those $\Gamma \in \mathbb{T}$ such that the arrow $\Gamma \to 1$ is representable;
- we define $(-)^{\mathcal{I}(\mathbb{T})}$ to be the composite

$$\mathbb{T} \xrightarrow{よ} [\mathbb{T}^{\mathsf{op}}, \mathbf{Set}] \to [\mathcal{I}(\mathbb{T})^{\mathsf{op}}, \mathbf{Set}].$$

Given a model $\mathcal{S}$ of $\mathbb{T}$, we have a functor

$$\begin{array}{ccc} \mathcal{I}(\mathbb{T}) & \overset{\mathsf{F}}{\dashrightarrow} & \mathcal{S} \\ \big\downarrow \int & \cong & \big\downarrow \int よ \\ \mathbb{T} & \underset{(-)^{\mathcal{S}}}{\longrightarrow} & [\mathcal{S}^{\mathsf{op}}, \mathbf{Set}] \end{array}$$

and F can be extended to a morphism of models of $\mathbb{T}$.