A Model of Type Theory with Directed Univalence in Bicubical Sets

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Directed Type Theory

Directed Type Theory

- Riehl-Shulman defines a type theory for ∞-categories with a model bisimplicial sets

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    1. Begin with HoTT

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  4. Predicate isCov(B : A → U) for covariant discrete fibrations

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  3. ∞-categories (Segal types) and univalent ∞-category (Rezk types) given internally as predicates on types
  4. Predicate isCov($B : A \to U$) for covariant discrete fibrations
  5. Cavallo, Riehl and Sattler have also (externally) defined the universe of covariant fibrations (the ∞-category of spaces and continuous functions) and shown *Directed Univalence:* $\mathrm{Hom}_{UCov}\, A\; B \simeq A \to B$

Constructive(?) Directed Type Theory

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- Can we make this constructive?

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    1. Begin with Cubical Type Theory

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  3. Use LOPS to define universe of covariant fibrations and construct directed univalence internally...

Constructive(?) Directed Type Theory

- Can we make this constructive?
  1. Begin with Cubical Type Theory
  2. Use a second cubical interval to define Hom-types
  3. Use LOPS to define universe of covariant fibrations and construct directed univalence internally...
     - ...unfortunately, directed univalence is a bit trickier than expected

Let's see how far the techniques from cubical type theory get us!

Defining Bicubical Directed Type Theory

Cubical Type Theory
(in the style of Orton-Pitts)

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$$\frac{x : \mathbb{2} \quad y : \mathbb{2}}{x \wedge y : \mathbb{2}} \qquad \frac{x : \mathbb{2} \quad y : \mathbb{2}}{x \vee y : \mathbb{2}}$$

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and equations...

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Directed Type Theory

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and equations...

i.e. generators for the Dedekind cubes

The Directed Interval

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  $x \leq y := x = x \wedge y$

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  - $p : \mathbb{I} \to \mathbb{2}$ is constant ($\Pi\ x\ y : \mathbb{I},\ p\ x = p\ y$)

The Directed Interval

- Why Dedekind cubes instead of Cartesian?
  $x \leq y := x = x \wedge y$

- We also add the following axioms:

  - $p : \mathbb{I} \to \mathbb{2}$ is constant (Π x y : $\mathbb{I}$, p x = p y)

  - $p : \mathbb{2} \to \mathbb{2}$ is monotone (Π x y : $\mathbb{2}$, if $x \leq y$ then $p\ x \leq p\ y$)

Defining Bicubical Directed Type Theory

Cubical Type Theory
(in the style of Orton-Pitts)

1. Begin with MLTT
2. Add an interval: $\mathbb{I}$
3. Specify gen. cofibrations for $\mathbb{I}$
4. Define filling problem for Kan fibrations
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Directed Type Theory

1. Begin with Cubical Type Theory
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Defining Bicubical Directed Type Theory

Cubical Type Theory
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Directed Type Theory

3. Specify gen. cofibrations for $\mathbb{2}$

Defining Bicubical Directed Type Theory

Cubical Type Theory
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3. Specify gen. cofibrations for $\mathbb{I}$

$$\frac{}{\mathsf{isCof} : \Omega \to \Omega}$$

$\mathsf{Cof} := \Sigma\, \phi : \Omega \,.\, \mathsf{isCof}\ \phi$

Directed Type Theory

3. Specify gen. cofibrations for $\mathbb{2}$

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$$\frac{}{\mathrm{isCof} : \Omega \to \Omega}$$

Cof := Σ ϕ : Ω . isCof ϕ

Cof closed under _∧_, _∨_, ⊥, ⊤

Directed Type Theory

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$\mathrm{Cof} := \Sigma\, \phi : \Omega \,.\, \mathrm{isCof}\ \phi$

Cof closed under $\_\wedge\_$, $\_\vee\_$, $\bot$, $\top$

$$\frac{x : \mathbb{I} \quad y : \mathbb{I}}{\_ : \mathrm{isCof}\ (x = y)}$$

$$\frac{\phi : \mathbb{I} \to \mathrm{Cof}}{\_ : \mathrm{isCof}\ (\Pi\, x : \mathbb{I} \,.\, \phi\ x)}$$

Directed Type Theory

3. Specify gen. cofibrations for $\mathbb{2}$

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Defining Bicubical Directed Type Theory

Cubical Type Theory
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4. Define filling problem for Kan fibrations

Directed Type Theory

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Defining Bicubical Directed Type Theory

Cubical Type Theory
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4. Define filling problem for Kan fibrations

hasCom : ($\mathbb{I}$ → U) → U
hasCom A = Π i j : $\mathbb{I}$.
Π α : Cof .
Π t : (Π x : $\mathbb{I}$. α → A x)
Π b : (A i)[α ↦ t i] .
(A j)[α ↦ t j; i = j ↦ b]

relCom : (A : U) → (A → U) → U
relCom A B = Π p : $\mathbb{I}$ → A .
hasCom (B ∘ p)

Directed Type Theory

4. Define filling problem for covariant fibrations

Defining Bicubical Directed Type Theory

Cubical Type Theory
(in the style of Orton-Pitts)

4. Define filling problem for Kan fibrations

$$
\begin{aligned}
&\mathrm{hasCom} : (\mathbb{I} \to U) \to U \\
&\mathrm{hasCom}\ A = \Pi\ i\ j : \mathbb{I}\ . \\
&\qquad \Pi\ \alpha : \mathrm{Cof}\ . \\
&\qquad \Pi\ t : (\Pi\ x : \mathbb{I}\ .\ \alpha \to A\ x) \\
&\qquad \Pi\ b : (A\ i)[\alpha \mapsto t\ i]\ . \\
&\qquad (A\ j)[\alpha \mapsto t\ j;\ i = j \mapsto b]
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{relCom} : (A : U) \to (A \to U) \to U \\
&\mathrm{relCom}\ A\ B = \Pi\ p : \mathbb{I} \to A\ . \\
&\qquad \mathrm{hasCom}\ (B \circ p)
\end{aligned}
$$

Directed Type Theory

4. Define filling problem for covariant fibrations

$$
\begin{aligned}
&\mathrm{hasCov} : (\mathbb{2} \to U) \to U \\
&\mathrm{hasCov}\ A = \Pi\ \alpha : \mathrm{Cof}\ . \\
&\qquad \Pi\ t : (\Pi\ x : \mathbb{2}\ .\ \alpha \to A\ x) \\
&\qquad \Pi\ b : (A\ \mathbb{0}_{\mathbb{2}})[\alpha \mapsto t\ \mathbb{0}_{\mathbb{2}}]\ . \\
&\qquad (A\ \mathbb{1}_{\mathbb{2}})[\alpha \mapsto t\ \mathbb{1}_{\mathbb{2}}]
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{relCov} : (A : U) \to (A \to U) \to U \\
&\mathrm{relCov}\ A\ B = \Pi\ p : \mathbb{2} \to A\ . \\
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5. Define universe of Kan fibrations

Directed Type Theory

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Defining Bicubical Directed Type Theory

Cubical Type Theory
(in the style of Orton-Pitts)

5. Define universe of Kan fibrations

- U_{Kan} given by LOPS construction for relCom

Directed Type Theory

5. Define universe of covariant fibrations

Defining Bicubical Directed Type Theory

Cubical Type Theory
(in the style of Orton-Pitts)

5. Define universe of Kan fibrations

- U_{Kan} given by LOPS construction for relCom

Directed Type Theory

5. Define universe of covariant fibrations

- U_{Cov} given by LOPS construction for relCov. **Lemma**: relCov is in UKan

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Cubical Type Theory
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Defining Bicubical Directed Type Theory

Cubical Type Theory *(in the style of Orton-Pitts)*	**Directed Type Theory**
6. Construct univalence	**6. Construct directed univalence**
• Key Idea: Glue type to attach equivalences to path structure	

Defining Bicubical Directed Type Theory

Cubical Type Theory
(in the style of Orton-Pitts)

6. Construct univalence

- Key Idea: Glue type to attach equivalences to path structure

Directed Type Theory

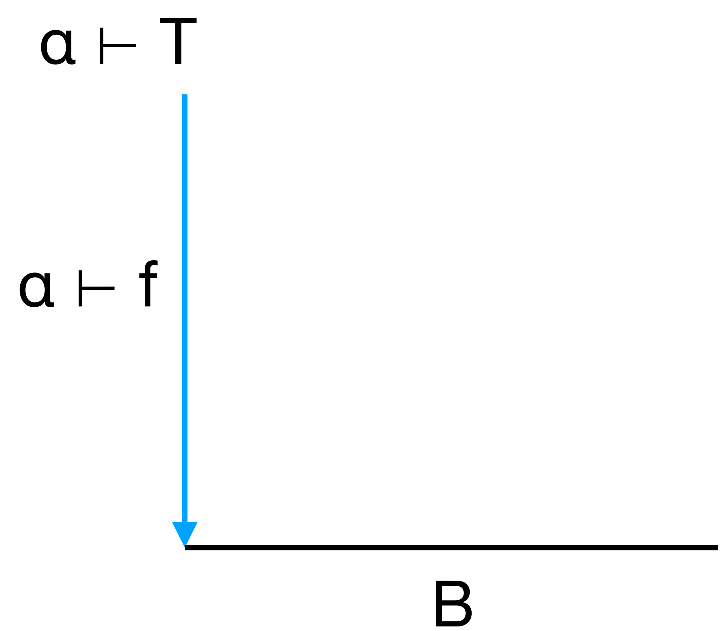
6. Construct directed univalence

- Key Idea: Glue type to attach ***functions*** to morphism structure

Glue Types

Glue Types

$\mathrm{Glue}\ [\ \alpha \mapsto (T\ ,\ f)\ ]\ B :=$



Glue Types

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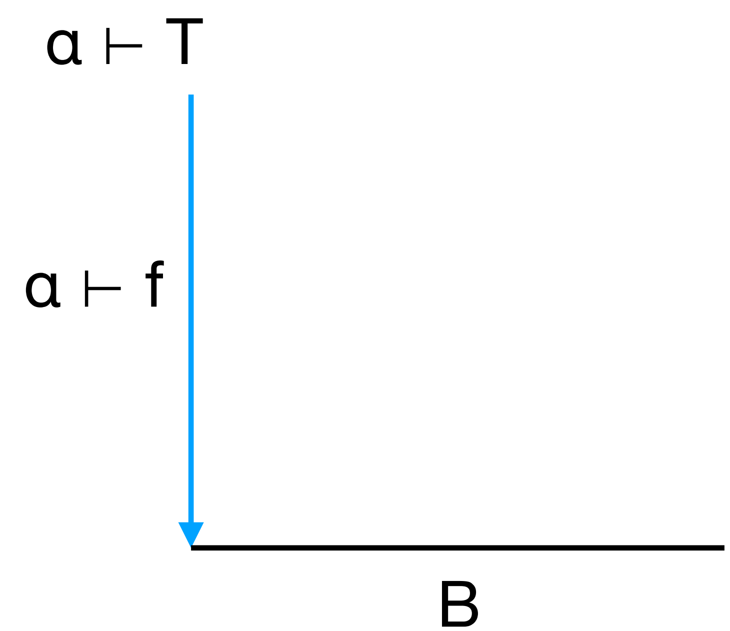

$\alpha \vdash$ Glue [$\alpha \mapsto (T , f)$] B $\equiv$ T

Glue Types

$\alpha \vdash T$

$\mathrm{Glue}\ [\ \alpha \mapsto (T\ ,\ f)\ ]\ B :=$ $\quad \alpha \vdash f$

B

$\alpha \vdash \mathrm{Glue}\ [\ \alpha \mapsto (T\ ,\ f)\ ]\ B \equiv T$

$\alpha \vdash t : T$

$\mathrm{glue}\ t\ b :=$ $\quad \alpha \vdash f$

$b : B$

Glue Types

$\alpha \vdash T$

Glue [$\alpha \mapsto (T , f)$] B :=

$\alpha \vdash f$

B

$\alpha \vdash \mathrm{Glue}\ [\ \alpha \mapsto (T , f)\ ]\ B \equiv T$

$\alpha \vdash t : T$

glue t b :=

$\alpha \vdash f$

b : B

$$\frac{g : \mathrm{Glue}\ [\ \alpha \mapsto (T , f)\ ]\ B}{\mathrm{unglue}\ g : B}$$

Glue Types

$\alpha \vdash T$

$\mathrm{Glue}\ [\ \alpha \mapsto (T\ ,\ f)\ ]\ B :=$

$\alpha \vdash f$

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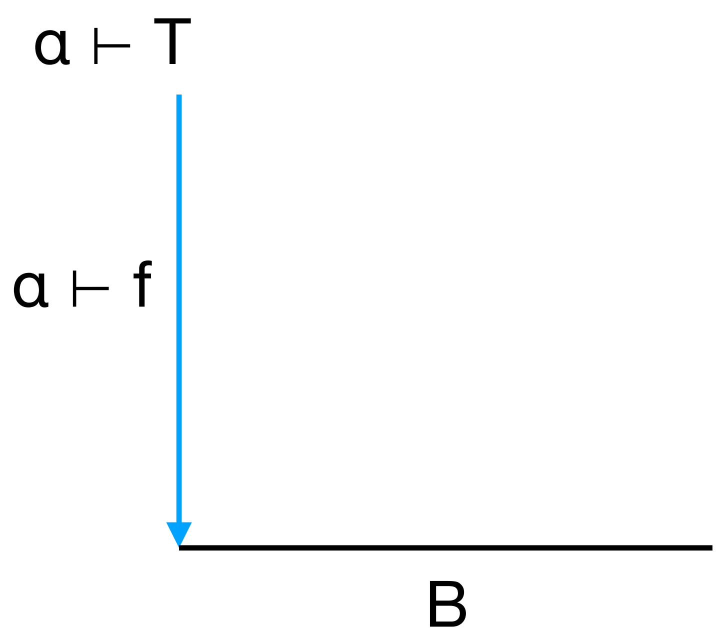

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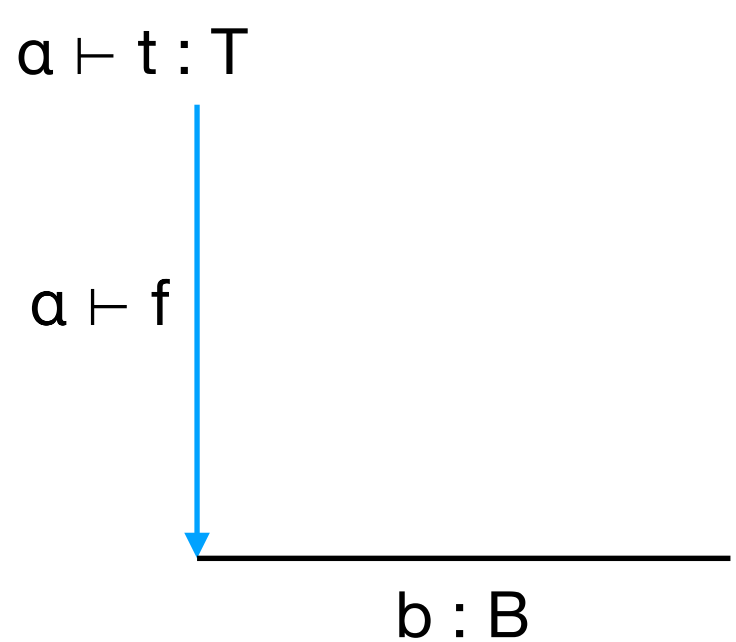

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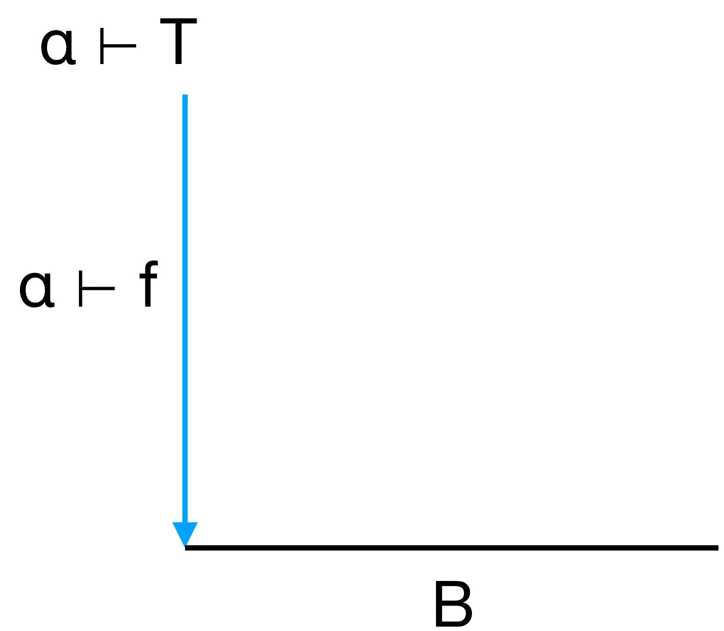

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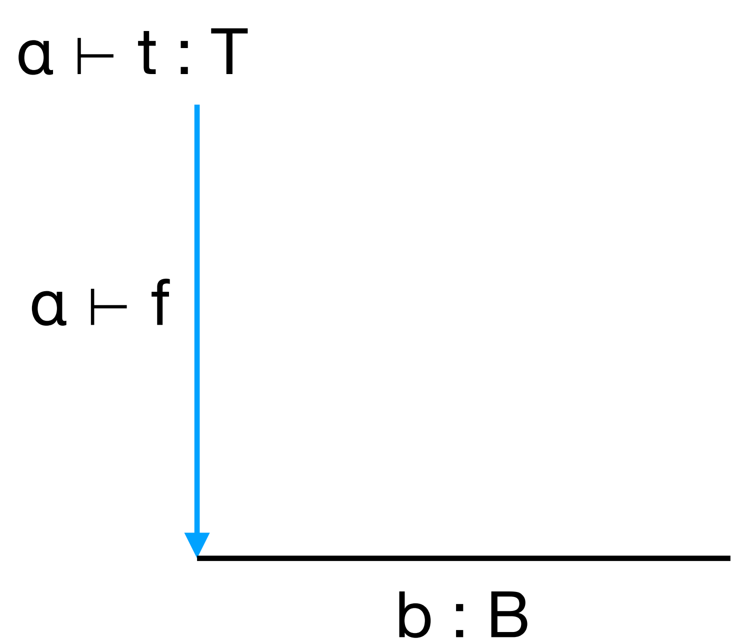

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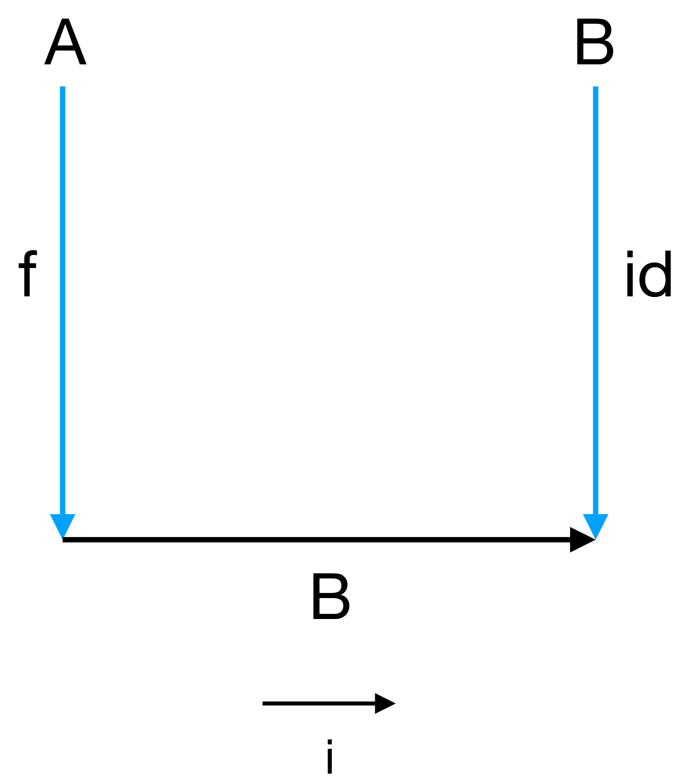
Defining
Directed Univalence

dua i A B f := Glue [i = 0 ↦ (A , f : A → B)
, i = 1 ↦ (B , id)] B : Hom_U A B



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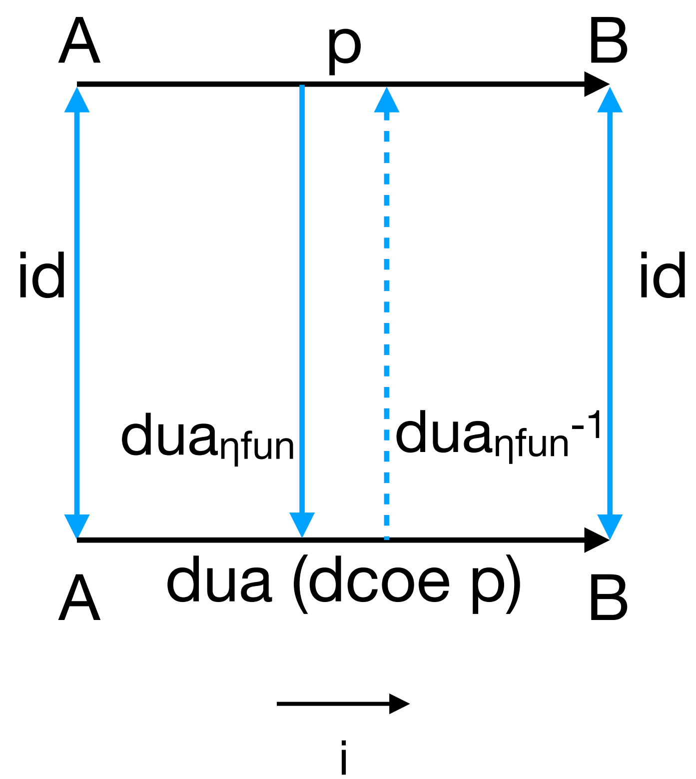

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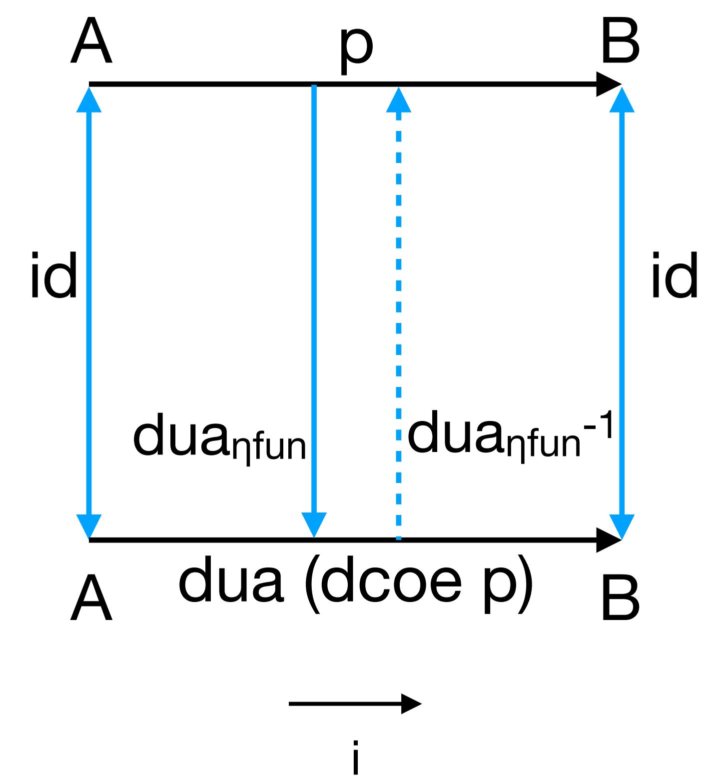

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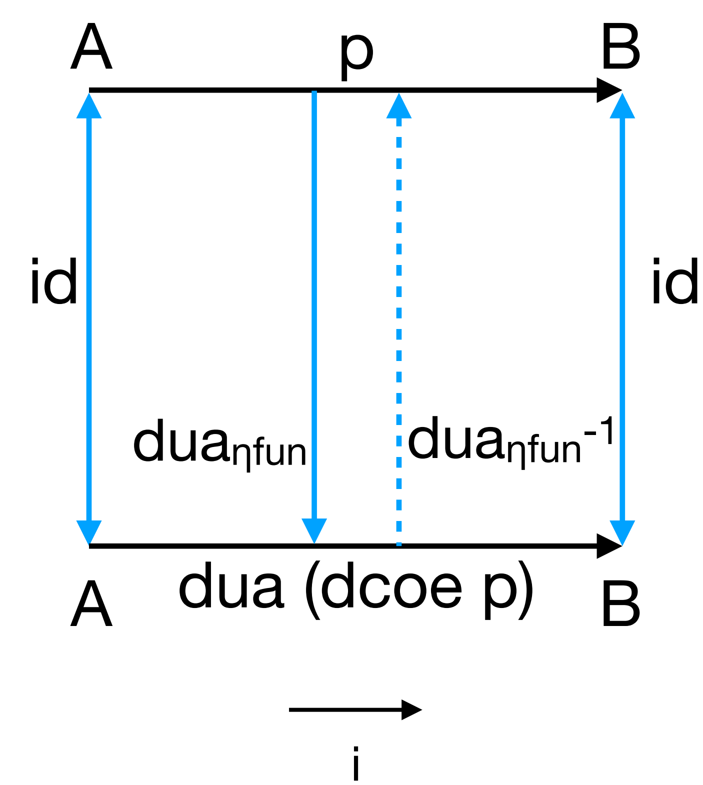

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- Agda: https://github.com/dlicata335/cart-cube

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- Note: We would love any/all feedback on the math that follows.

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- (informal/incomplete) Definition: A *generalized Reedy category* is a category C along with a degree function $\delta : \mathrm{ob}\, C \to \mathbb{N}$ such that every morphism (that isn't an iso) factors through an object of strictly smaller degree

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The Dedekind Cubes

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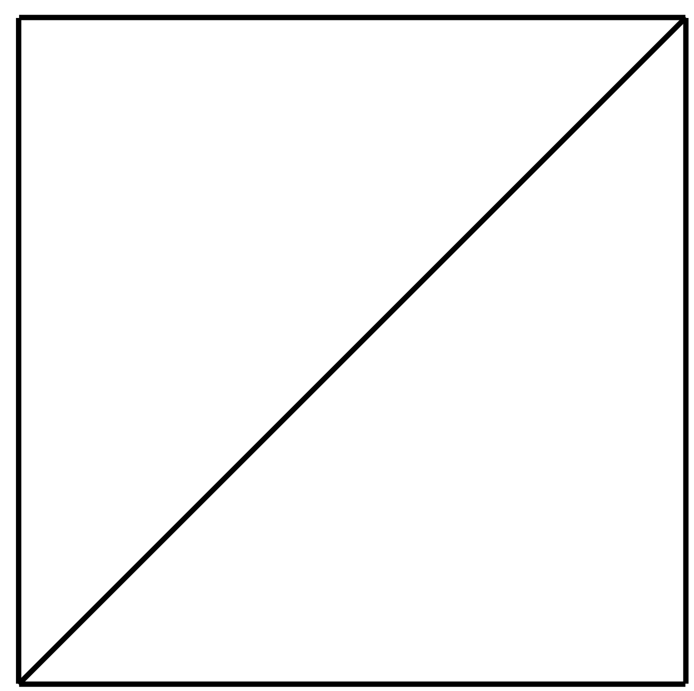

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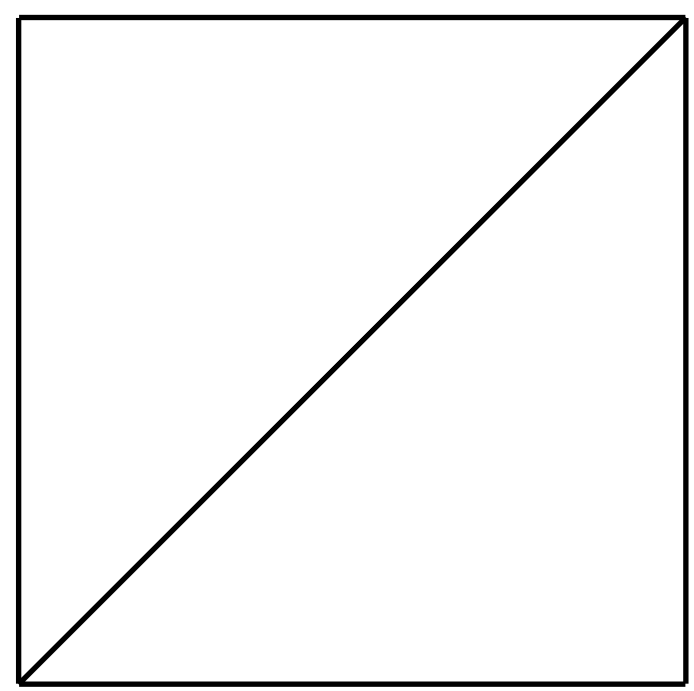

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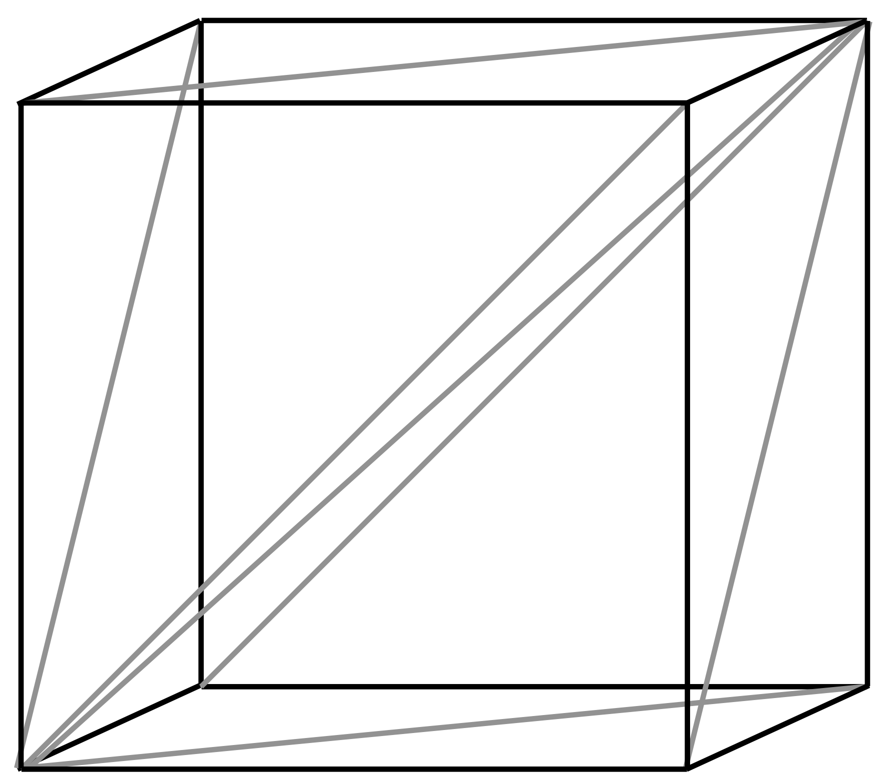

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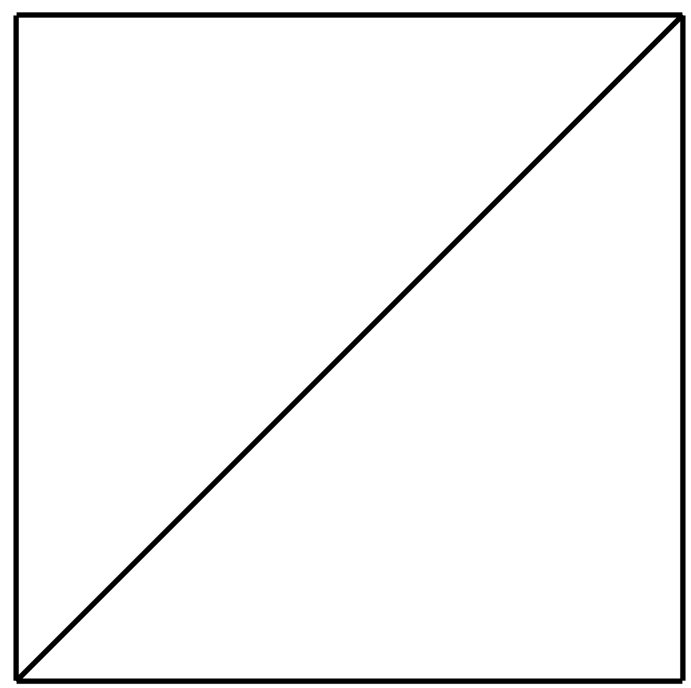

down by a connection

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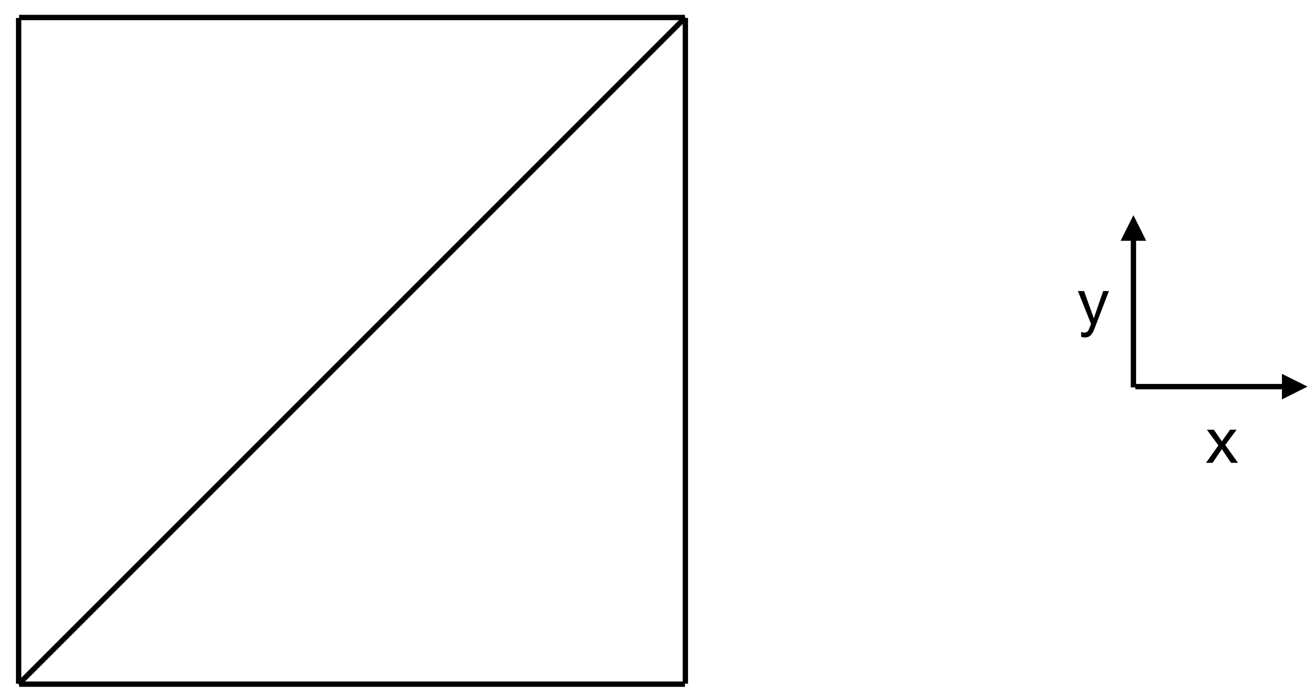

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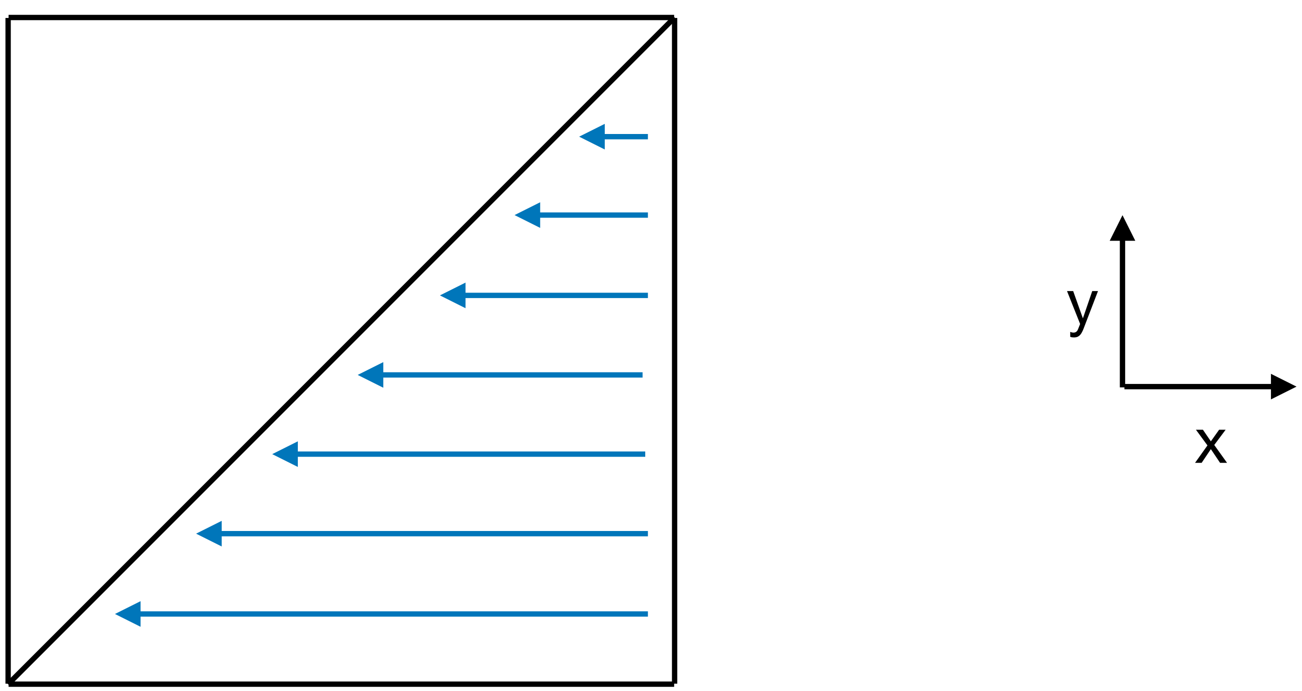

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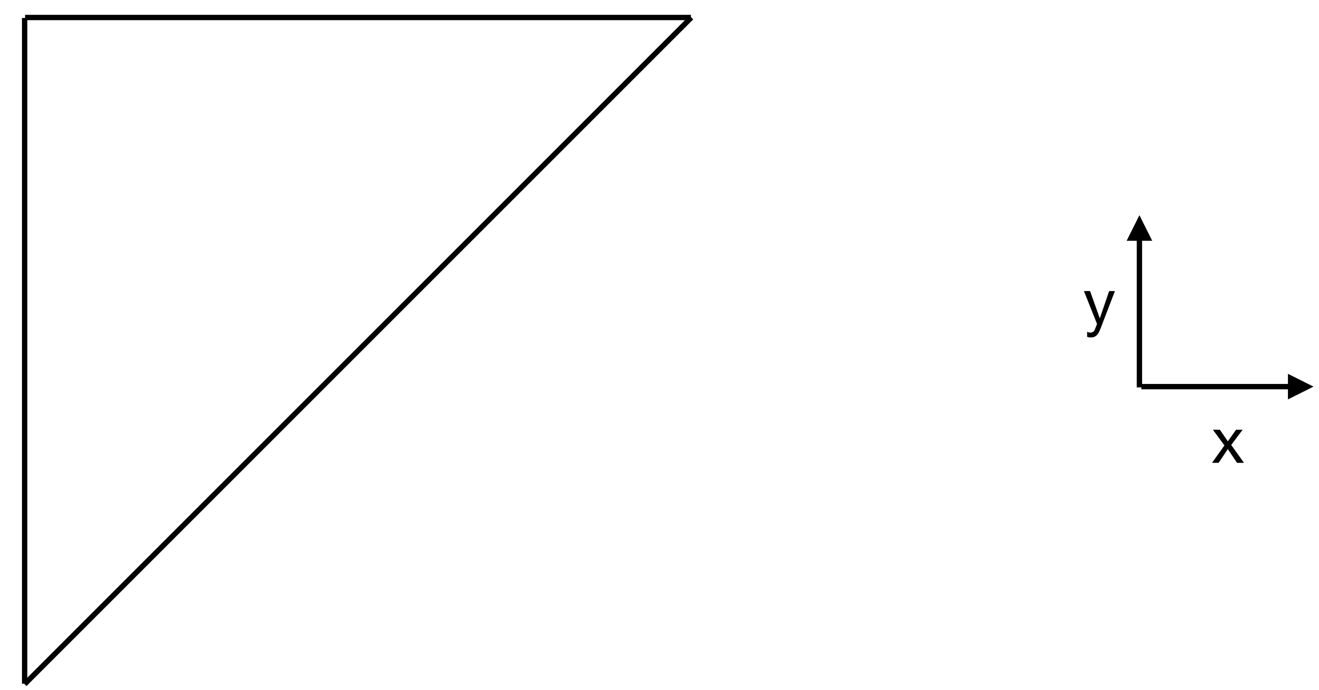

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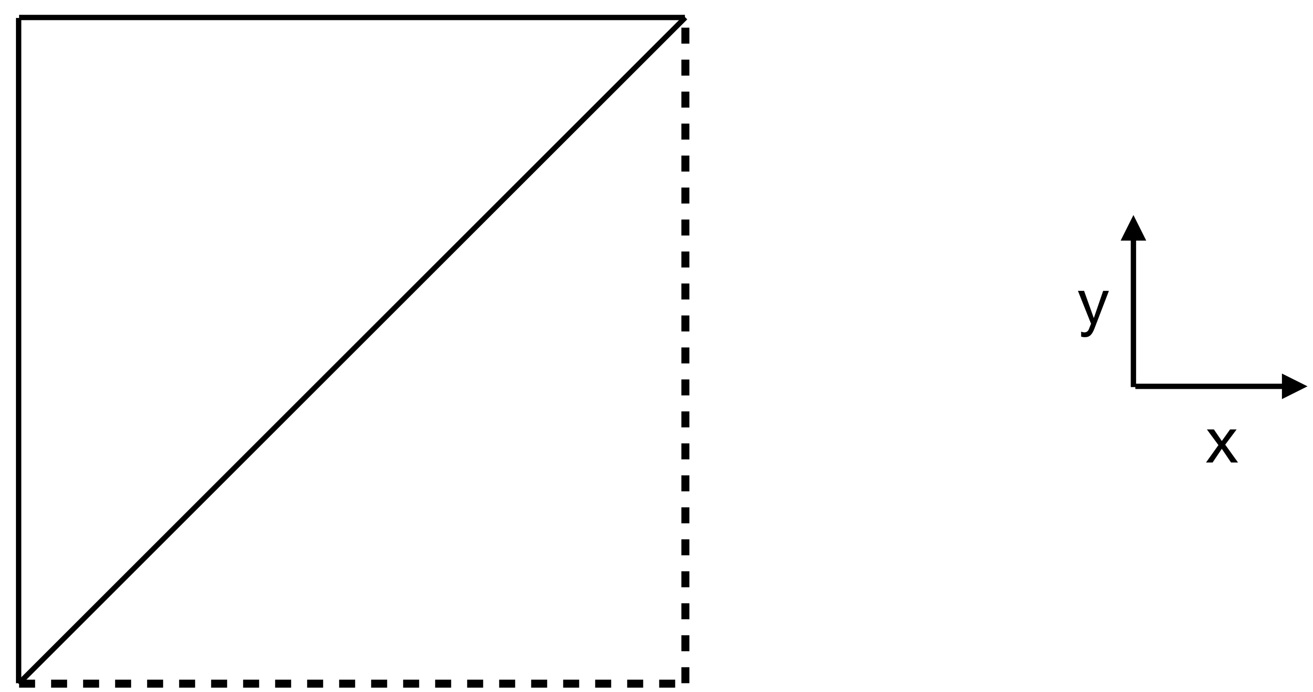

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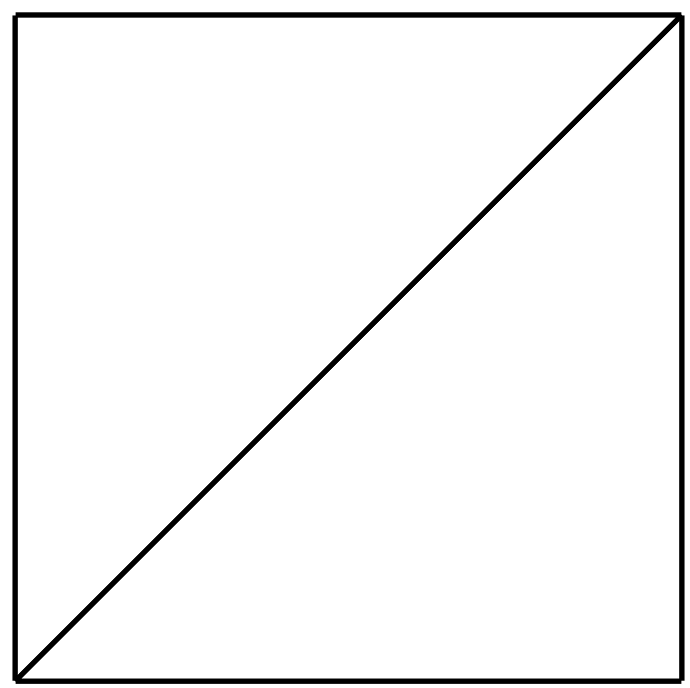

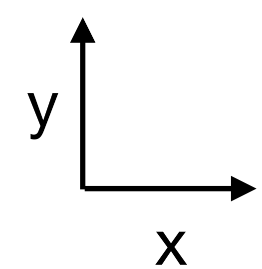

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  - The Comparison Lemma: [SGA 4, The Elephant]

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- Lemma (with Christian Sattler): The prism category is equivalent to the full subcategory of simplicial sets containing subobjects of the Dedekind cubes (Γ , ϕ) generated by the following formulae:
  - $\top$: true
  - $x \leq y$: the equalizer of the degeneracy map x and connection $x \wedge y$
  - $\phi \wedge \psi$: the pullback of the subobjects (Γ, ϕ) and (Γ, ψ)
  - $\phi \vee \psi$: the pushout of the pullback for $(\Gamma, \phi \wedge \psi)$

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- Question: For which categories C is Im(C) Reedy?

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- Can we make this even more cubical?

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