Type-theoretic modalities for synthetic $(\infty, 1)$-categories

Ulrik Buchholtz & Jonathan Weinberger

TU Darmstadt

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2 Model in simplicial spaces (inside cubical spaces)

3 Modalities from shape operations

4 Right adjoint types

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Outline

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Foundations: Synthetic \((\infty, 1)\)-categories à la Riehl–Shulman I

In order to develop *synthetic higher category theory*, Riehl and Shulman introduced a *Simplicial Type Theory* (STT) in [RS17]: MLTT with additional layers of shapes, allowing for defining *synthetic* \((\infty, 1)\)-categories as *complete Segal/Rezk types*.

As a main feature, STT postulates *extension types* (after Lumsdaine–Shulman), i.e. for shape inclusions \(\Phi \hookrightarrow \Psi\), families \(A : \Psi \to \mathcal{U}\), and partial sections \(a : \prod_{t : \Phi} A(t)\) there exists the type of sections

\[
\left\langle \prod_{t : \Psi} A(t) \middle|_{\Phi} \right\rangle \triangleq \left\{ \begin{array}{l}
\Phi \\
\downarrow \\
\Psi
\end{array} \xrightarrow{a} A
\right\}
\]

*judgmentally extending* \(a\).

**Example & Definition:** For a type \(A\) and terms \(x, y : A\), define the *hom-type*

\[
\text{hom}_A(x, y) := \left\langle \Delta^1 \to A \right|_{[x, y]} \right\rangle.
\]
Foundations: Synthetic \((\infty, 1)\)-categories à la Riehl–Shulman II

Definitions from [RS17]:

- A type \(A\) is a **Segal type** if \((\Delta^2 \to A) \simto (\Lambda_1^2 \to A)\) (Joyal).
- A Segal type \(A\) is a **Rezk type** if \(\text{idtoiso}_A : \prod_{x,y:A} \text{Id}_A(x, y) \simto \text{iso}_A(x, y)\).
- A type \(A\) is a **discrete type** if \(\text{idtorarr}_A : \prod_{x,y:A} \text{Id}_A(x, y) \simto \text{hom}_A(x, y)\).

These notions coincide with their classical analogues in the intended semantics in (a model structure representing) the \(\infty\)-topos of simplicial spaces, \(\text{PSh}_\infty(\Delta)\).

**Goal:** Extend the \(\infty\)-category theory developed in [RS17]. Namely, add universes, other notions of fibrations, and the traditional Yoneda embedding \(\text{y}_A : A \to (A^{\text{op}} \to \text{Space})\).

Besides Riehl–Shulman’s work, we heavily rely on Licata–Shulman–Riley’s modal framework, cf. Dan’s recent talk! For related work in bicubical sets, cf. Matt’s upcoming talk!
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Simplicial spaces inside cubical space

The category $sSet$ of simplicial sets is the category of presheaves on the category $\Delta$ of finite ordinals with monotone maps as morphisms.

The category $cSet$ of cubical sets is the category of presheaves on the category of powers of the ordinal 2 with monotone maps as morphisms.

We want to define universes internally which, due to [LOPS18] becomes possible using tinyness of the cubical interval $\square^1$.

Simplicial sets form an essential subtopos of cubical sets.

This has been discussed by Sattler [Sat18], Kapulkin–Voevodsky [KV18], and Streicher-W [SW18].

One can show that this lifts to the level of $\infty$-toposes. Since this constitutes a topological modality sheafification becomes an internal operation ([RSS17]) which by the theory of compact types treated in [Rij18] can be expressed in rather elementary terms.
Universes of simplicial types

Start with a \textit{strict} universe in cubical spaces [Shu19]. From this we derive:

• \textbf{Simp}: universe of simplicial types since we have a topological modality [RSS17]
• \textbf{Cat}: universe of (complete) Segal types due to our new notion of \textit{cocartesian family}
• \textbf{Space}: universe of discrete types due to Riehl–Shulman’s notion of covariant family
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Modalities: $♭$ and $^{\text{op}}$

However, the universes constructed this way are classifying only for the cohesively discrete (crisp) types.

We also want to have opposite categories $A^{^{\text{op}}}$. Hence, we introduce the modalities $♭$ and $^{\text{op}}$ as in the framework of Licata–Shulman–Riley to this theory.

We have a mode theory with $c$ (cohesive/cubical) and $x : c \vdash f(x) : c$ (flat) as well as $x : c \vdash o(x) : c$ (representing the opposite cubical type/category) with equations

$$f(f(x)) = f(x), \quad o(f(x)) = f(x), \quad o(o(x)) = x, \quad f(o(x)) = f(x),$$

and

$$f(x \times y) = f(x) \times f(y), \quad o(x \times y) = o(x) \times o(y).$$
Operations on topes and shapes I

**Problem:** In order to get the Yoneda embedding, we need to get $\text{hom}_A(a, b)$ for $a : A^{\text{op}}$ and $b : A$ (for $A :: \text{Cat}$).

**Solution:** Instead of ordinary $\text{hom}$-types construct a covariant fibration $\text{Tw}(A) \to A^{\text{op}} \times A$ and obtain the “hom type” as the fiber. Here, $\text{Tw}(A)$ is the *twisted arrow type* (traditionally, the category of elements of the uncurried Yoneda embedding) with 0-simplices

$$a \xrightarrow{f} b$$

and 1-simplices:

$$\begin{array}{c}
\varphi_0 \\
\uparrow \\
\downarrow \\
\varphi_1
\end{array}
\quad
\begin{array}{c}
\varphi_0 \\
\uparrow \\
\downarrow \\
\varphi_1
\end{array}
\quad
\begin{array}{c}
\varphi_0 \\
\uparrow \\
\downarrow \\
\varphi_1
\end{array}
\quad
\begin{array}{c}
\varphi_0 \\
\uparrow \\
\downarrow \\
\varphi_1
\end{array}
$$

Classically, the twisted arrow space is defined by reindexing along the functor

$\varepsilon := \text{op} * \text{id} : \triangle \to \triangle$. This does not yield an extension type in Riehl–Shulman’s sense.
We get the twisted arrow type using right adjoint types ($U$-types) in the sense of [LRS19].

First, we axiomatize operations on topes and shapes according to

\[
\frac{f : \Xi \to \Xi'}{\Xi | \Phi \vdash \Phi' \ f} \quad \text{and} \quad \frac{F \text{ oper} \ \{\Xi \mid \varphi\} \ \text{shape}}{F\{\Xi \mid \varphi\} \ \text{shape}}.
\]

Defining opposites and join for topes, we can then lift these to the level of shapes as

\[
\{I \mid \varphi\} \ast \{J \mid \psi\} := \{I + 1 + J \mid \varphi \ast \psi\}, \quad \{I \mid \varphi\}^\text{op} = \{I \mid \varphi^\text{op}\}.
\]

From this, we can define $\varepsilon := \text{op} \ast \text{id}$ for shapes. Unary operations induce modalities on the base category, hence we can define the twisted arrow types as $U$-types w.r.t. to $\varepsilon$. 
After the work of Licata–Shulman–Riley consider a type theory fibered over a type theory of modes:

Given a shape $\Phi$ and an arbitrary mode context $\gamma$, we get a universe $\gamma \vdash c_\Phi$ of small types over $\Phi$.

For any small type $\gamma \vdash n : c_\Phi$ there is a small type $\gamma \vdash T(\Phi)(n)$ of contexts over $n$, a *comprehension object* in the sense of [LRS19].

Endomorphisms $f : \square \to \square$ give rise to mode morphisms $n : c_\Phi \vdash f n : c_{f\Phi}$.
Some rules of the type theory of modes I

\[ \begin{align*}
\gamma \vdash \\
\frac{\gamma \vdash \gamma \vdash n : c_\Phi}{\gamma, x : T(\Phi)(n) \vdash} \\
\gamma \vdash a \\
\frac{\gamma \vdash (\Phi \text{ shape})}{\gamma \vdash c_\Phi} \\
\frac{\gamma \vdash \gamma \vdash n : c_\Phi}{\gamma \vdash T(\Phi)(n)} \\
\gamma \vdash n : a \\
\frac{\gamma \vdash \gamma \vdash \gamma \vdash n : c_\Phi}{\gamma \vdash \gamma \vdash \gamma \vdash m : T(\Phi)(n)} \\
\frac{\gamma \vdash \gamma \vdash \gamma \vdash n : c_\Phi}{\gamma \vdash \gamma \vdash \gamma \vdash 1 : T(\Phi)(n)} \\
\frac{\gamma \vdash \gamma \vdash \gamma \vdash n : c_\Phi \quad f \text{ op}_1}{\gamma \vdash \gamma \vdash \gamma \vdash f_0 : c_\Phi} \\
\frac{\gamma \vdash \gamma \vdash \gamma \vdash n : c_\Phi \quad \gamma \vdash \gamma \vdash \gamma \vdash m : T(\Phi)(n)}{\gamma \vdash f(m) : T(\Phi)(fn)} \\
\frac{\gamma \vdash \gamma \vdash \gamma \vdash \gamma \vdash f_0 : c_\Phi \quad \gamma \vdash \gamma \vdash \gamma \vdash m : T(\Phi)(n)}{f(n.m) \equiv f(n). f(m)}
\end{align*} \]
Some rules of the type theory of modes II

\[
\begin{align*}
\gamma \vdash n \Rightarrow m : a \\
\gamma \vdash n : c_\Phi \\
\gamma \vdash 1 \Rightarrow f(1) : T(f \Phi)(f n)
\end{align*}
\]
Some rules of the type theory–on–top

\[ \Gamma \vdash \gamma \]

\[ \Gamma \vdash T(\Phi)(n) \ A \quad \gamma \vdash n : c_\Phi \]

\[ \Gamma, x : A \vdash_{\gamma,x:T(\Phi)(n)} \]

\[ \Gamma \vdash \gamma \vdash a \ A \]

\[ \Gamma \vdash a \ A_1 \quad \Gamma \vdash a \ A_2 \]

\[ \Gamma \vdash a \ A_1 + A_2 \]

\[ \Gamma \vdash a \ A_1 \times A_2 \]

\[ \Gamma \vdash_{\gamma} n : a \ N : A \]

\[ \Gamma \vdash T(\Phi)(n) \ A \quad \gamma \vdash n : c_\Phi \]

\[ \Gamma, x : A \vdash_{\gamma,x:T(\Phi)(n)} x : T(\Phi)(n) \ A \]
Semantics of the fibrational framework I

- Mode contexts $\gamma$ are (modeled as) toposes (with sufficient homotopical/logical structure).
- Modes–in–context $\gamma \vdash a$ are geometric morphisms $E \to [\gamma \vdash]$.
- Types–over–modes $\Gamma \vdash_{\gamma}$ are objects of $[\gamma \vdash]$.
- Terms–over–mode terms $\Gamma \vdash_{\gamma-a} A$ are objects of the fibers $E_{[\Gamma]}$.

\[
\begin{array}{ccc}
\mathcal{E} & \rightarrow
\end{array}
\]

\[
\begin{array}{ccc}
\Gamma \vdash_{\gamma-a} A & \in \mathcal{E}_{[\Gamma]} & s([\Gamma \vdash_{\gamma-a}]) \xrightarrow{[\Gamma \vdash_{\gamma-n:a} N: A]} [\Gamma \vdash_{\gamma-a} A]
\end{array}
\]

- The empty mode context $\cdot \vdash$ is the terminal topos.
Semantics of the fibrational framework II

- Universes $\gamma \vdash c$ are projections

\[
\begin{array}{c}
\left[\gamma\right] \times \mathcal{E} \xrightarrow{\left[\gamma \vdash c\right]} \left[\gamma\right]
\end{array}
\]

with canonical section $\left[\gamma \vdash \emptyset : c\right] = \lambda X. \langle X, 1 \rangle$.

- Comprehension objects $\gamma \vdash T(n)$ are interpreted by Artin glueing of $\left[\gamma \vdash n : a\right]$:

\[
\begin{array}{c}
\left[\gamma, x : T(n) \vdash\right] \xrightarrow{} \mathcal{E} \xrightarrow{\text{cod}} \\
\left[\gamma \vdash T(n)\right] \downarrow \quad \downarrow \text{cod} \\
\left[\gamma \vdash\right] \xrightarrow{\left[\gamma \vdash n : a\right]} \mathcal{E}
\end{array}
\]

In particular, in our intended model of cubical spaces mode contexts will be of the form $c\text{Sp} / \Phi$ for a shape $\Phi$. 
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Right adjoint types I

- Endomorphisms $f : □ \to □$ give rise to adjoint pairs $f^*: \text{cSp}/\Phi \leftrightarrow \text{cSp}/\Phi : f_!, f_! \dashv f^*$.
- The functor $f_!$ (on the level of modes) corresponds to mode morphisms.
- $f^*$ gives rise to right adjoint types, aka $U$-types.
- We get a bijection

$$\begin{align*}
\left\{ \Gamma \vdash f(k): T(f\Phi)(f \ n) \ a : A \right\} & \leftrightarrow_{1:1} \left\{ \Gamma \vdash k: T(\Phi)(n) \ b : U f A \right\}.
\end{align*}$$
Right adjoint types II

\[
\frac{\Gamma \vdash T(f\Phi)(f\, n) \quad \gamma \vdash n : c_{\Phi}}{\Gamma \vdash T(\Phi)(f\, n) \quad \gamma \vdash n : c_{\Phi}} \quad \text{U-Form}
\]

\[
\frac{\Gamma \vdash f(k) : T(f\Phi)(f\, n) \quad \gamma \vdash n : c_{\Phi} \quad \gamma \vdash k : T(\Phi)(n)}{\Gamma \vdash f(k) : T(\Phi)(n) \quad \lambda f M : U_f A \quad \text{U-Intro}}
\]

\[
\frac{\Gamma \vdash k : T(\Phi)(n) \quad N : U_f A \quad \gamma \vdash n : c_{\Phi}}{\Gamma \vdash f(k) : T(f\Phi)(f\, n) \quad N()_f : A \quad \text{U-Elim}}
\]

\[
\frac{\Gamma \vdash f(k) : T(f\Phi)(f\, n) \quad N()_f : A \quad \lambda f N()_f \equiv N \quad \lambda f M()_f \equiv M}{\lambda f N()_f \equiv N \quad \lambda f M()_f \equiv M}
\]
One can show that the action of the mode morphism when forming a $U$-type builds upon the structure of a dependent right adjoint, cf. [BCM18] et al., 2018:

Assume an operation $f : \Phi \to \Psi$, inducing $f! \dashv f^* : E_\Phi \to E_\Psi$.

For $[\Gamma] \in [\gamma]$ and $[n] : [\gamma] \to E_\Phi$, consider $[A]$ and $[k]$ as in:

Then we get a correspondence:

$$f_!( [k] [\Gamma] ) \dashv \dashv [A] \quad \quad [k] [\Gamma] \dashv \dashv U_f [A] \dashv \dashv f^* ([A])$$
Externally, the twisted arrow simplicial space is constructed by reindexing along the functor $\varepsilon := \text{op} \ast \text{id}$. Thus, we internalize it by considering the $U$-type w.r.t. the endofunctor $\varepsilon$. Note that there are two natural transformations $\eta_0 : \text{op} \Rightarrow \varepsilon \Leftarrow \text{id} : \eta_1$ in particular, for any shape $\Phi$ giving rise to a diagram:

$$
\begin{array}{ccc}
\Box/\Phi & \xrightarrow{\varepsilon!} & \Box/\varepsilon \Phi \\
\downarrow \text{op}! & \quad & \downarrow \eta_0 \Phi \\
\Box/\text{op} \Phi & \xrightarrow{} & \Box/\varepsilon \Phi
\end{array}
$$
Twisted arrow types II

\[
\Gamma \vdash T(\varepsilon \Phi)(\varepsilon n) \quad A \\
\Gamma \vdash k : T(\Phi)(n) \\
A \quad \gamma \vdash k : T(\Phi)(n) \quad \gamma \vdash n : c_\Phi
\]
\[\text{tw-Form}\]
\[
\Gamma \vdash k : T(\Phi)(n) \\
tw_A^k(a_0, a_1)
\]
\[
(\lambda^{\eta_0} \Phi a)^\text{op} \equiv a_0 \\
\lambda^{\eta_1} \Phi a \equiv a_1 \\
\Gamma \vdash k : T(\Phi)(n) \\
\gamma \vdash k : T(\Phi)(n) \quad \gamma \vdash n : c_\Phi
\]
\[\text{tw-Intro}\]
\[
\Gamma \vdash k : T(\Phi)(n) \\
\lambda^{\text{tw}} a : tw_A^k(a_0, a_1)
\]
\[
\Gamma \vdash k : T(\Phi)(n) \\
b : tw_A^k(a_0, a_1) \\
\gamma \vdash k : T(\Phi)(n) \quad \gamma \vdash n : c_\Phi
\]
\[\text{tw-Elim}\]
\[
(\lambda^{\eta_0} \Phi b(\_\text{tw})^\text{op} \equiv a_0 \\
\lambda^{\eta_1} \Phi b(\_\text{tw} \equiv a_1 \\
\lambda^{\text{tw}} a(\_\text{tw} \equiv a \\
\lambda^{\text{tw}} b(\_\text{tw} \equiv b
Using that the flat modality can be defined as the $U$-type w.r.t. the terminal projection functor $!: 1 \to 1$ one can show for crisp Segal types $A$ that e.g. $\flat \text{hom}_A(a_0, a_1) \simeq \flat \text{tw}_A(a_0, a_1)$ using the ensuing computation rules for $U$-types.
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Perspectives

Work in progress:

• Give a full proof of an analog of the “classical Yoneda Lemma” using twisted arrow types.
• Define fibrancy structures internally on the universes $\text{Simp}$, $\text{Cat}$, and $\text{Space}$, possibly à la Orton (PhD thesis).
• Do we get (enough of) the expected 2-dimensional structure for the theory of Segal/Rezk types, cf. Riehl–Shulman, Riehl–Verity’s $\infty$-cosmos theory?

Based on the same frameworks:

• Cavallo–Riehl–Sattler: Directed univalence for simplicial type theory [CRS18]
• Licata–Weaver: Directed univalence for bicubical directed type theory [LW18]

Selection of further work on directed type theory:

• Altenkirch–Sestini: “Naturality for free”, 2019)
• Cavallo–Harper: parametric CTT, 2019)
• North: directed HoTT & wfs, 2018/19
• Nuyts: directed HoTT, 2015+; w/ Devriese: Menkar, ultimode presheaf proof assistant https://github.com/anuyts/menkar
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D. Licata, M. Weaver (2018): Directed univalence in bicubical directed type theory Presentation at MURI Meeting, Pittsburgh

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E. Riehl, M. Shulman (2017): A type theory for synthetic $\infty$-categories

E. Riehl, D. Verity (2019): Elements of $\infty$-Category Theory
Book in progress

E. Rijke, M. Shulman, B. Spitters (2017): Modalities in homotopy type theory
arXiv:1706.07526

E. Rijke (2018): Classifying types
PhD thesis, CMU

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arXiv:1805.04126

M. Shulman (2019): All $(\infty, 1)$-toposes have strict univalent universes
arXiv:1904.07004

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Thank you!