Monoidal Model Categories and Cubical Homotopy Theory

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Joint with Donald Yau

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My background

Homotopy theorist (student of Hovey; professor in Ohio).
Good at model categories; novice in type theory.

Want to do work that is useful; happy to get involved with projects coming out of HoTT.

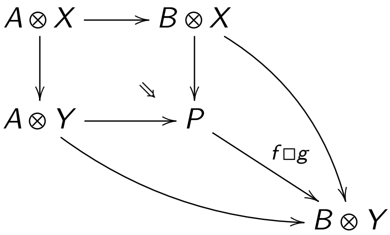
Keywords: model categories, Quillen equivalences, Bousfield localization, operads, Goodwillie calculus, equivariant/motivic homotopy, homological algebra, representation theory, left/right induced model structures, Grothendieck universes, locally presentable categories, ...

As Paul Erdös said "My brain is open" - feel free to email me.

Monoidal Model Categories (implies Ho(M) monoidal)

$(\mathrm{M}, \otimes, \mathbb{1})$ is a closed symmetric monoidal model category.

The *pushout product* of $f : A \longrightarrow B$ and $g : X \longrightarrow Y$, is the corner map:



This is a monoidal product on the arrow category $\overrightarrow{\mathrm{M}}^{\square}$.

Pushout Product Axiom: If f and g are cofibrations, so is $f \square g$. If either is also a weak equivalence, so is $f \square g$.

Main result

The *projective model structure* on $\overrightarrow{\mathrm{M}}^{\square}$ has weak equivalences and fibrations defined levelwise.

Theorem (W.-Yau; arXiv:1703.05359; Math Scandinavica 2018)

If M *is a monoidal model category, then so are* $\overrightarrow{\mathrm{M}}^{\square}$, $\mathrm{M}^{I^{\times n}}$, *and* $\mathrm{M}^{\square^{op}}$.

$$\begin{array}{ccc} V_0 & \xrightarrow{\alpha_0} & W_0 \\ {\scriptstyle f_V}\downarrow & & \downarrow{\scriptstyle f_W} \\ V_1 & \xrightarrow{\alpha_1} & W_1 \end{array} \qquad \begin{array}{ccc} X_0 & \xrightarrow{\beta_0} & Y_0 \\ {\scriptstyle f_X}\downarrow & & \downarrow{\scriptstyle f_Y} \\ X_1 & \xrightarrow{\beta_1} & Y_1 \end{array}$$

The pushout product in $\overrightarrow{\mathrm{M}}^{\square}$ is the map

$$(f_W \square f_X) \coprod_{f_V \square f_X} (f_V \square f_Y) \xrightarrow{\alpha \square_2 \beta} f_W \square f_Y$$

History and applications

First proved by Hovey (unpublished) for M cofibrantly generated.
Pavlov-Scholbach 2018 inductive proof.
PP axiom in injective or Reedy: see Barwick 2010 or White 2017.

Examples of monoidal model categories: Set, Top, $sSet$, $sMod_R$, spectra (symmetric, orthogonal, S-modules), equivariant/motivic, $Ch(R)$, StMod$(k[G])$, Cat, Groupoids, ...

Our main application: Smith $\mathcal{O}$-ideals = algebras over operad $\overrightarrow{\mathcal{O}}^{\square}$ in $\overrightarrow{\mathrm{M}}^{\square}$. We provide a Quillen equivalence

$$\left\{\text{Smith } \mathcal{O}\text{-Ideals}\right\} \underset{\text{ker}}{\overset{\text{coker}}{\rightleftarrows}} \left\{\mathcal{O}\text{-Algebra Maps}\right\}$$

Connection to Homotopy Type Theory

Pushout product axiom gives better understanding of (trivial) cofibrations.

Techniques to decompose higher pushout products, e.g. $\alpha \,\square_2\, \beta$.

Characterization of projective cofibrations (dual to Mike's talk).

Don't need $\otimes$ to be Cartesian, or cofibrations = monomorphisms.

Related to Thierry's model structure on $Set^{(C\times\square)^{op}}$ (e.g. $sSet^{\square^{op}}$).

Techniques could work on prismatic cubical sets (Matt's talk). Or bicubical sets based on model categories of cubical sets presented by Emily and Steve (BCH, ABCFHL, CCHM, ACCRS), when monoidal/substructural.

Main result

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Proof of main theorem (1)

We'll focus on $\overrightarrow{\mathrm{M}}^{\square}$. To get $\mathrm{M}^{I^{\times n}}$ and $\mathrm{M}^{\square^{op}}$, iterate.

To save space, write $W_1 X_0$ for $W_1 \otimes X_0$, etc. Let $f_V : V_0 \longrightarrow V_1$, and f_W, f_X, f_Y similarly.

If $\alpha : f_V \longrightarrow f_W$ and $\beta : f_X \longrightarrow f_Y$, then $\alpha \,\square_2\, \beta$ is the following commutative square in M:

$$
\begin{array}{ccc}
\left(W_1X_0 \coprod_{W_0X_0} W_0X_1\right) \coprod_{\left(V_1X_0 \coprod_{V_0X_0} V_0X_1\right)} \left(V_1Y_0 \coprod_{V_0Y_0} V_0Y_1\right) & \xrightarrow{\ \zeta\ } & W_1Y_0 \coprod_{W_0Y_0} W_0Y_1 \\
\Big\downarrow {\scriptstyle (f_W \square f_X) \coprod_{f_V \square f_X} (f_V \square f_Y)} & & \Big\downarrow {\scriptstyle f_W \square f_Y} \\
W_1X_1 \coprod_{V_1X_1} V_1Y_1 & \xrightarrow{\ \alpha_1 \square \beta_1\ } & W_1Y_1
\end{array}
$$

Proof (cont)

Lemma (Hovey): In $\overrightarrow{\mathrm{M}}^{\square}$, γ from $f : X_0 \longrightarrow X_1$ to $g : Y_0 \longrightarrow Y_1$ is a (trivial) cofibration iff γ_0 and $\gamma_1 \circledast g : X_1 \coprod_{X_0} Y_0 \longrightarrow Y_1$ are.

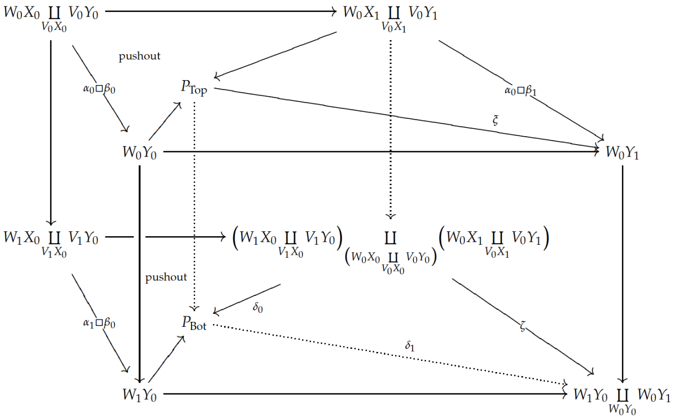
Assume α is a cofibration and β is a (trivial) cofibration in $\overrightarrow{\mathrm{M}}^{\square}$

We must prove ζ is a (trivial) cofibration and the pushout corner map

$$\Big(W_1 X_1 \coprod_{V_1 X_1} V_1 Y_1\Big) \coprod_{Z} \Big(W_1 Y_0 \coprod_{W_0 Y_0} W_0 Y_1\Big) \xrightarrow{(\alpha_1 \square \beta_1) \circledast (f_W \square f_Y)} W_1 Y_1$$

is a (trivial) cofibration.

Proof (cont)

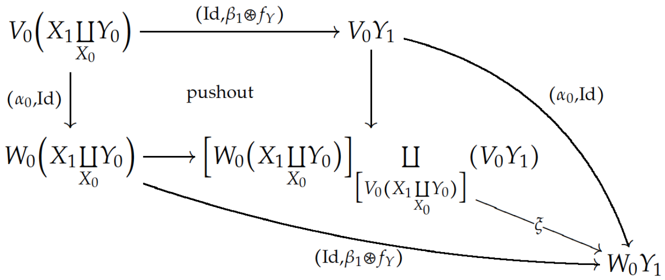


$W_0X_0 \coprod_{V_0X_0} V_0Y_0$
$W_0X_1 \coprod_{V_0X_1} V_0Y_1$
pushout
$\alpha_0\Box\beta_0$
P_{Top}
$\alpha_0\Box\beta_1$
ξ
W_0Y_0
W_0Y_1
$W_1X_0 \coprod_{V_1X_0} V_1Y_0$
$\left(W_1X_0 \coprod_{V_1X_0} V_1Y_0\right) \coprod_{\left(W_0X_0 \coprod_{V_0X_0} V_0Y_0\right)} \left(W_0X_1 \coprod_{V_0X_1} V_0Y_1\right)$
pushout
δ_0
$\alpha_1\Box\beta_0$
P_{Bot}
δ_1
ζ
W_1Y_0
$W_1Y_0 \coprod_{W_0Y_0} W_0Y_1$

Proof (cont)

$\zeta = \delta_1 \circ \delta_0$, and δ_0 is a pushout of $\alpha_1 \square \beta_0$ so is a (trivial) cofibration.

δ_1 is a pushout of ξ, which we rewrite as the pushout product $\alpha_0 \square (\beta_1 \circledast f_Y)$, below, so both are (trivial) cofibrations.



Proof (cont)

To finish, rewrite

$$\Big(W_1X_1 \coprod_{V_1X_1} V_1Y_1\Big)\coprod_Z\Big(W_1Y_0 \coprod_{W_0Y_0} W_0Y_1\Big) \xrightarrow{(\alpha_1\square\beta_1)\otimes(f_W\square f_Y)} W_1Y_1$$

as:

$$\begin{array}{ccc}
\Big(W_1X_1 \coprod_{V_1X_1} V_1Y_1\Big)\coprod_Z\Big(W_1Y_0 \coprod_{W_0Y_0} W_0Y_1\Big) & \xrightarrow{(\alpha_1\square\beta_1)\otimes(f_W\square f_Y)} & W_1Y_1 \\
\downarrow\cong & & \downarrow = \\
W_1\Big(X_1\coprod_{X_0}Y_0\Big) \coprod_{\left(V_1\coprod_{V_0}W_0\right)\left(X_1\coprod_{X_0}Y_0\right)} \Big(V_1\coprod_{V_0}W_0\Big)Y_1 & \xrightarrow{(\alpha_1\otimes f_W)\square(\beta_1\otimes f_Y)} & W_1Y_1
\end{array}$$

Recap

Theorem (W.-Yau; arXiv:1703.05359; Math Scandinavica 2018)

If M *is a monoidal model category, then so are* $\overrightarrow{\mathsf{M}}^{\square}$, $\mathsf{M}^{I^{\times n}}$, *and* $\mathsf{M}^{\square^{op}}$.

Lemma (Hovey): In $\overrightarrow{\mathsf{M}}^{\square}$, γ from $f : X_0 \longrightarrow X_1$ to $g : Y_0 \longrightarrow Y_1$ is a (trivial) cofibration iff γ_0 and $\gamma_1 \circledast g : X_1 \coprod_{X_0} Y_0 \longrightarrow Y_1$ are.

If α is cof and β is (triv) cof, then let $\gamma = \alpha \square_2 \beta$.

We proved $\gamma_0 = \zeta$ and $\gamma_1 \circledast g$ from previous slide, are (triv) cof's.

A word on cofibrant generation

Hovey's proof used that, if M is cof. gen., it's sufficient to check $I \square I \subset$ Cofibrations and $I \square J \subset$ Triv. Cofibrations.

Lots of monoidal non-cof. gen. model categories:

1. Christensen-Hovey: absolute model structure on $\mathrm{Ch}(\mathbb{Z})$.
2. Barthel-May-Riehl: r-model structure on $dgRmod_r$.
3. Adamek-Herrlich-Rosicky-Tholen model on Cat.
4. Strom model on compactly generated spaces.
5. $\mathrm{Pro}(C)$ where C is a tensor model category.

References

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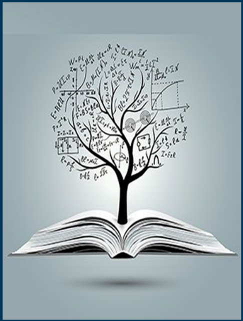

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