Natural Model Semantics of Comonadic Modal Type Theory

Colin Zwanziger

Department of Philosophy
Carnegie Mellon University

August 15, 2019
at the International Conference on Homotopy Type Theory
Carnegie Mellon University

Problem: Semantics of Comonadic Type Theory

Comonads are pervasive. So **comonadic dependent type theory** (NPP 2008, Shulman 2018) has many intended models, *e.g.*:

- (Higher) Grothendieck toposes + $\Delta\Gamma$ (Shulman 2018, 2019)
- In particular, cubical sets + the 0-skeleton (LOPS 2018)
- Groupoids + discretization (*cf.* Zwanziger 2018)

What about a general **categorical semantics** for comonadic DTT?

A Solution: Morphisms of Natural Models

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- Simple picture: the comonadic operator is interpreted as a morphism of models of DTT that is a comonad
- I will work with morphisms of *natural* models.
- Natural models are a nice categorical characterization of categories with families (CwFs) (Awodey 2012, 2018, Fiore 2012)
- The relevant morphisms of natural models and CwFs were developed by Newstead (2018) and BCMMPS (2018), respectively.

Morphism Semantics to Date

- BCMMPS use morphisms of CwFs to interpret DTT with an endo-adjunction.
- In Zwanziger (2019): morphisms of natural models for DTT with an adjunction.
- Same approach for comonadic DTT presently.

So morphisms of NMs/CwFs have a broader applicability than the comonadic case.

Outline

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Natural Models

Definition (Awodey, Fiore 2012)

A **natural model** *consists of*

- *a category* C
- *a distinguished terminal object* $1 \in C$
- *presheaves* $\mathrm{Ty}, \mathrm{Tm} : C^{op} \to Set$
- *a representable natural transformation* $p : \mathrm{Tm} \to \mathrm{Ty}$

Conventions

Convention

- *An object* $\Gamma \in C$ *is a "context".*
- *An element* $A \in \mathrm{Ty}(\Gamma)$ *is a "type in context* Γ*".*
- *An element* $a \in \mathrm{Tm}(\Gamma)$ *such that* $p_\Gamma(a) = A$ *is a "term* a *of type* A *in context* Γ*".*

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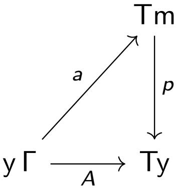
This last is represented by the following commutative diagram:



Below, as here, we will freely use the Yoneda lemma to identify presheaf elements $x \in P(C)$ with the corresponding map $x : \mathrm{y}\, C \to P$.

Comprehension as Representability

Representability of $p : \mathsf{Tm} \to \mathsf{Ty}$ means the following:

Definition

Given a context $\Gamma \in \mathcal{C}$ *and a type* $A \in \mathsf{Ty}(\Gamma)$ *in the context* Γ*, there is* $\Gamma.A \in \mathcal{C}$, $p_A : \Gamma.A \to \Gamma$, *and* $v_A : \mathsf{y}(\Gamma.A) \to \mathsf{Tm}$ *such that the following diagram is a pullback:*

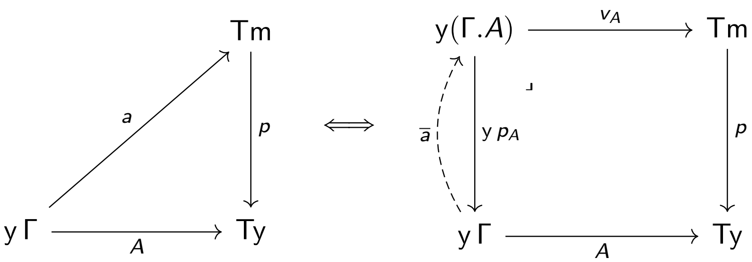
$$
\begin{array}{ccc}
\mathsf{y}(\Gamma.A) & \xrightarrow{v_A} & \mathsf{Tm} \\
{\scriptstyle \mathsf{y}\,p_A}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\
\mathsf{y}\,\Gamma & \xrightarrow[A]{} & \mathsf{Ty}
\end{array}
$$

These $\Gamma.A, p_A, v_A$ *constitute the* **comprehension** *of* A*.*

Terms *vs.* Sections

Remark

Terms are interchangeable with a "comprehension" as sections, as depicted by the following:



See Awodey (2018) for more on natural models.

Lax Morphisms

Definition

A **lax morphism** *of natural models $F : C \to D$ consists of:*

- *a functor, also denoted $F : C \to D$, between the underlying categories*
- *a natural transformation $\phi_{\mathrm{Ty}} : F_! \, \mathrm{Ty}_C \to \mathrm{Ty}_D$*
- *a natural transformation $\phi_{\mathrm{Tm}} : F_! \, \mathrm{Tm}_C \to \mathrm{Tm}_D$*

such that the following diagram commutes:

$$\begin{array}{ccc} F_! \, \mathrm{Tm}_C & \xrightarrow{\phi_{\mathrm{Tm}}} & \mathrm{Tm}_D \\ {\scriptstyle F_! p_C} \downarrow & & \downarrow {\scriptstyle p_D} \\ F_! \, \mathrm{Ty}_C & \xrightarrow[\phi_{\mathrm{Ty}}]{} & \mathrm{Ty}_D \end{array}$$

The definitions of this section are essentially those of Newstead (2018).

Notation

Convention

Given a lax morphism $F : C \to D$, and a type $A \in \mathrm{Ty}(\Gamma)$ in context $\Gamma \in C$, we write F/A for the composite

$$\mathrm{y}\, F\Gamma \cong F_!\, \mathrm{y}\, \Gamma \xrightarrow{F_! A} F_!\, \mathrm{Ty}_C \xrightarrow{\phi_{\mathrm{Ty}}} \mathrm{Ty}_D$$

Similarly, given a term $a \in \mathrm{Tm}(\Gamma)$, we write F/a for the composite

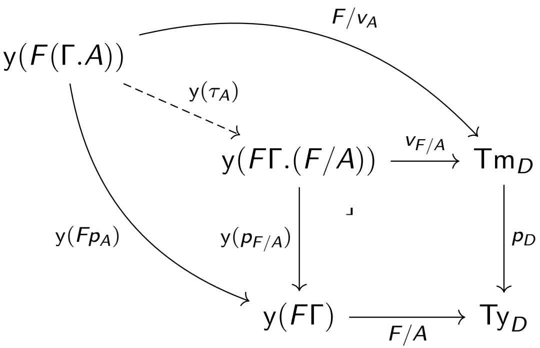
$$\mathrm{y}\, F\Gamma \cong F_!\, \mathrm{y}\, \Gamma \xrightarrow{F_! a} F_!\, \mathrm{Tm}_C \xrightarrow{\phi_{\mathrm{Tm}}} \mathrm{Tm}_D$$

One may think of F/A and F/a as the results of applying the morphism F to A and a. These operations are implicated in the interpretation of (respectively) formation and introduction rules for modal type operators.

Lax Preservation of Context Extension

Remark

Let $F : C \to D$ be a lax morphism. Then, given a type $A \in \mathrm{Ty}_C(\Gamma)$ in context $\Gamma \in C$, there is a unique comparison map $\tau_A : F(\Gamma.A) \to F\Gamma.(F/A)$ such that $Fp_A = p_{F/A} \circ \tau_A$ and $F/v_A = v_{F/A} \circ \mathrm{y}(\tau_A)$, i.e., such that the following diagram commutes:



Morphisms

Definition

Let $F : C \to D$ be a lax morphism. Then F is said to preserve context extension if, for each type $A \in \mathrm{Ty}_C(\Gamma)$ in each context $\Gamma \in C$, the comparison map $\tau_A : F(\Gamma.A) \to F(\Gamma).(F/A)$ is an isomorphism.

Definition

A lax morphism $F : C \to D$ of natural models that preserves context extension and terminal objects is called a **morphism of natural models***.*

CoTT

Contexts and Judgments

We will use the comonadic fragment of Shulman (2018)'s real-cohesive type theory.

We have two variable judgments, denoted

$$u :: A$$

and

$$x : A \quad ,$$

and the typing judgement has form

$$u_1 :: A_1, \ldots, u_m :: A_m \mid x_1 : B_1, \ldots, x_n : B_n \vdash t : C \quad .$$

CoTT

Contexts and Judgments (cont'd)

The two variable judgements lead to a duplication of the context and variable rules:

$$\frac{}{\cdot \mid \cdot \vdash}\ \text{Emp.}$$

$$\frac{\Delta \mid \cdot \vdash B\ \mathit{type}}{\Delta, u :: B \mid \cdot \vdash}\ \text{Ext.}^{\flat} \qquad \frac{\Delta, u :: A, \Delta' \mid \Gamma \vdash}{\Delta, u :: A, \Delta' \mid \Gamma \vdash u : A}\ \text{Var.}^{\flat}$$

CoTT

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$$\frac{\Delta \mid \Gamma \vdash B \ \mathit{type}}{\Delta \mid \Gamma, x : B \vdash} \text{Ext.} \qquad \frac{\Delta \mid \Gamma, x : A, \Gamma' \vdash}{\Delta \mid \Gamma, x : A, \Gamma' \vdash x : A} \text{Var.}$$

CoTT

The Comonad $\flat$

$$\frac{\Delta \mid \cdot \vdash B \ \mathit{type}}{\Delta \mid \Gamma \vdash \flat B \ \mathit{type}}\ \flat\text{-Form.} \qquad \frac{\Delta \mid \cdot \vdash t : B}{\Delta \mid \Gamma \vdash t^{\flat} : \flat B}\ \flat\text{-Intro.}$$

CoTT

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$$\frac{\Delta \mid \Gamma \vdash s : \flat A \qquad \begin{array}{c} \Delta \mid \Gamma, x : \flat A \vdash B \ \mathit{type} \\ \Delta, u :: A \mid \Gamma \vdash t : B[u^{\flat}/x] \end{array}}{\Delta \mid \Gamma \vdash (\mathit{let}\ u^{\flat} := s\ \mathit{in}\ t) : B[s/x]} \ \flat\text{-Elim.}$$

CoTT

The Comonad $\flat$ (Conversions)

$$\frac{\Delta \mid \cdot \vdash s : A \qquad \Delta, u :: A \mid \Gamma \vdash t : B[u^\flat/x] \qquad \Delta \mid \Gamma, x : \flat A \vdash B \ \mathit{type}}{\Delta \mid \Gamma \vdash (\mathit{let}\ u^\flat := s^\flat\ \mathit{in}\ t) \equiv t[s/u] : B[s^\flat/x]}\ \flat\text{-}\beta\text{-Conv.}$$

$$\frac{\Delta \mid \Gamma \vdash s : \flat A \qquad \Delta \mid \Gamma, x : \flat A \vdash t : B \qquad \Delta \mid \Gamma, x : \flat A \vdash B \ \mathit{type}}{\Delta \mid \Gamma \vdash \mathit{let}\ u^\flat := s\ \mathit{in}\ t[u^\flat/x] \equiv t[s/x] : B[s/x]}\ \flat\text{-}\eta\text{-Conv.}$$

Cartesian Comonads

Our notion of model for **CoTT** takes an appealingly simple form:

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Definition

Let $\flat : \mathcal{E} \to \mathcal{E}$ be an endomorphism of natural models on $\mathcal{E}$. This $\flat$ is said to be a **Cartesian comonad** *on $\mathcal{E}$ when its underlying functor is a comonad.*

The requirement that $\flat$ be a morphism of natural models is a preservation condition analogous to finite limit preservation in the topos semantics of modal logic (*cf.* Zwanziger 2017).

Notation

Some further notation:

We write $\mathcal{E}^\flat$ for the category of coalgebras for $\flat$, U or $(-)_0 : \mathcal{E}^\flat \to \mathcal{E}$ for the forgetful functor, and $K : \mathcal{E} \to \mathcal{E}^\flat$ for the cofree functor. As the name suggests, we have $U \dashv K$.

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Definition

Let $\Delta = (\Delta_0, \Delta_1 : \Delta_0 \to \flat\Delta_0) \in \mathcal{E}^\flat$. *Then*

- $\flat A :\equiv (\flat/A) \circ \mathrm{y}(\Delta_1) : \mathrm{y}(\Delta_0) \to \mathrm{Ty}$, *where* $A : \mathrm{y}(\Delta_0) \to \mathrm{Ty}$, *and*
- $\flat a :\equiv (\flat/a) \circ \mathrm{y}(\Delta_1) : \mathrm{y}(\Delta_0) \to \mathrm{Tm}$, *where* $a : \mathrm{y}(\Delta_0) \to \mathrm{Tm}$.

It is this new $\flat(-)$, not $\flat/(-)$, which will interpret the formation and introduction rules for the type operator $\flat$ of **CoTT**.

Interpretation

A context $\Delta \mid \Gamma$ will be interpreted not as an object of $\mathcal{E}$, but as an arrow $[\![\Delta \mid \Gamma]\!]$ with codomain a coalgebra. However, the interpretation of a type $\Delta \mid \Gamma \vdash A$ is simply in $\mathrm{Ty}(\mathrm{dom}[\![\Delta \mid \Gamma]\!])$.

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The partial interpretation function $[\![-]\!]$ is given by recursion on raw syntax as follows:

$$\text{(Ext.).}\quad [\![\Delta \mid \Gamma, x : B]\!] = [\![\Delta \mid \Gamma]\!] \circ p_{[\![B]\!]} \in \mathcal{E}/\,\mathrm{cod}[\![\Delta \mid \Gamma]\!]$$

$$\text{(Var.).}\quad [\![\Delta \mid \Gamma, x : A \vdash x : A]\!] = v_{[\![A]\!]} \in \mathrm{Tm}_{\mathcal{E}}(\mathrm{dom}[\![\Delta \mid \Gamma]\!].[\![A]\!])$$

Interpretation (continued)

In the special case of $\Delta \mid \cdot$, $[\![\Delta \mid \cdot]\!]$, abbreviated $[\![\Delta]\!]$, will be an identity.

(Emp.). $[\![\cdot]\!] = \mathrm{id}_{\flat 1} \in \mathcal{E}/UK1$

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(Using that $\flat$ is a morphism of NMs, $\mathrm{dom}[\![\Delta]\!].\flat[\![B]\!]$ admits a canonical coalgebra structure.)

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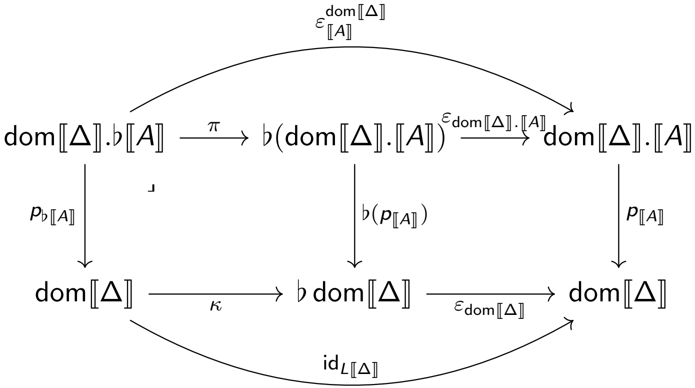
(Var.$^\flat$). $[\![\Delta, u :: A \Vdash u : A]\!] = v_{[\![A]\!]} \circ \mathsf{y}(\varepsilon_{[\![A]\!]}^{\mathsf{dom}[\![\Delta]\!]}) \in \mathsf{Tm}(\mathsf{dom}[\![\Delta, u :: A]\!])$
(See next slide.)

Interpretation (continued)

Here, $\varepsilon^{\mathrm{dom}[\![\Delta]\!]}_{[\![A]\!]} : \mathrm{dom}[\![\Delta]\!].\flat[\![A]\!] \to \mathrm{dom}[\![\Delta]\!].[\![A]\!]$ is the "indexed counit" induced over the coalgebra $(\mathrm{dom}[\![\Delta]\!], \kappa : \mathrm{dom}[\![\Delta]\!] \to \flat\,\mathrm{dom}[\![\Delta]\!])$:



The left-hand square exists and is a pullback because $\flat$ is a morphism (not just a lax morphism).

Interpretation (continued)

($\flat$-Form.). $[\![\Delta \mid \cdot \vdash \flat B]\!] = \flat[\![B]\!] \in \mathsf{Ty}(\mathsf{dom}[\![\Delta]\!])$

($\flat$-Intro.). $[\![\Delta \mid \cdot \vdash t^{\flat} : \flat B]\!] = \flat[\![t]\!] \in \mathsf{Tm}(\mathsf{dom}[\![\Delta]\!])$

($\flat$-Elim.). $[\![\Delta \mid \cdot \vdash (let\ u^{\flat} := r\ in\ t) : B[r/x]]\!] = [\![t]\!] \circ \mathsf{y}(\overline{[\![r]\!]}) \in \mathsf{Tm}(\mathsf{dom}[\![\Delta]\!])$

Result

Theorem

The interpretation $[\![-]\!]$ is sound. That is, it is defined on all derivable contexts, types, and terms, and, furthermore, all contexts, types, and terms identified by equations receive the same interpretation.

Conclusion

- We used morphisms of natural models to interpret comonadic DTT.
- This work captures the groupoid model and cubical sets, with the comonads indicated, and other 1-topos models.
- Approach generalizes to some other type theories (BCMMPS 2018, Zwanziger 2019), but how far can one push this (*cf.* LSR 2017)?

Thanks for your attention!