Uniform Kan fibrations in simplicial sets (jww Eric Faber)

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Warning

This is work in progress and not as thoroughly checked as I would have liked!

Section 1

Motivation

Voevodsky

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Constructs a model of univalent type theory in simplicial sets, interpreting the dependent types as Kan fibrations.

He builds on well-known properties of the Kan-Quillen model structure to prove such facts as:

Π-types

Let $f : Y \to X$ be a Kan fibration. If $f^* : sSets/X \to sSets/Y$ is pull back along f, then its right adjoint Π_f (push forward) preserves Kan fibrations.

This is important for interpreting the Π-types in type theory.

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Theorem (Bezem-Coquand-Parmann)

The classical result which says that if A and B are Kan, then so is A^B, is not constructively valid.

Some constructive results

Theorem (Henry)

One can use the standard definitions of a (trivial) Kan fibration to show *constructively* that there is a model structure on simplicial sets.

However, we cannot prove *constructively* that in this model structure every object is cofibrant (!).

Gambino-Henry show how this can be extended to a model of homotype type theory as well, modulo some issues:

- They only have a weak form of Π (constructively).
- An appropriate coherence theorem to turn this into a genuine model of type theory is (so far) missing.

I expect we will hear more about this in the next talk!

Our aim

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Define a fibration structure (that of a "uniform Kan fibration") in simplicial sets such that ...

- maps with this structure are closed under Π, constructively.
- every Kan fibration can be equipped with the structure of a uniform Kan fibration, classically.

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Objection

Wait, wasn't that already done by Gambino & Sattler in their paper "The Frobenius condition, right properness, and uniform fibrations"?

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Response

True, but they were unable to show constructively that their notion of uniform Kan fibration was *local* (the fibration structure is locally representable).

Local class

Definition

Let us say that a structure on morphisms is *local* if: to equip a morphism $f : Y \to X$ with this structure it is necessary and sufficient to do this for every pullback of f along a map $x : \Delta^n \to X$, in such a way our choices are stable under pulling back along maps $\alpha : \Delta^m \to \Delta^n$.

$$
\begin{tikzcd}
Y_{x \cdot \alpha} \arrow[r] \arrow[d] & Y_x \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
\Delta^m \arrow[r, "\alpha"'] & \Delta^n \arrow[r, "x"'] & X
\end{tikzcd}
$$

Because they were unable to show that their notion of a uniform Kan fibration is local, Gambino & Sattler were unable to show *constructively* that universal uniform Kan fibrations exist.

Our aim, again

Our aim

Define a notion of a uniform Kan fibration in simplicial sets such that ...

- they are closed under Π, constructively.
- every Kan fibration can be equipped with the structure of a uniform Kan fibration, classically.
- the structure of being a uniform Kan fibration is local.

Today, I will explain our definition as a modification of the one by Gambino & Sattler. So let's first recall that one.

Section 2

The definition

Cofibrations

Cofibrations (for us)

A map $f : Y \to X$ of simplicial sets is a *cofibration* if each $f_n : X_n \to Y_n$ is a complemented monomorphism.

Warning

These are not the cofibrations in the Henry model structure!

Note that cofibrations are closed under composition and stable under pullback.

Uniform Kan fibration *à la* Gambino-Sattler

Uniform Kan fibration *à la* Gambino-Sattler

A morphism $f : Y \to X$ is a *uniform Kan fibration* if for any commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "m"'] & Y^I \arrow[d, "(s/t, f^I)"] \\ B \arrow[r] \arrow[ur, dotted] & Y \times_X X^I \end{tikzcd}$$

in which m is a cofibration, there is a chosen filler (as indicated), in such a way that for any map $b : B' \to B$ the chosen fillers in

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "b^* m"'] & A \arrow[r] \arrow[d, "m"'] & Y^I \arrow[d, "(s/t, f^I)"] \\ B' \arrow[r, "b"'] \arrow[urr, dotted] & B \arrow[r] \arrow[ur, dotted] & Y \times_X X^I \end{tikzcd}$$

commute which each other. (Here $I = \Delta^1$.)

Simplicial Moore path object

In Van den Berg & Garner, we defined a simplicial Moore path functor. The idea is that there is an endofunctor M on simplicial sets together with natural transformations $r_X : X \to MX$, $s_X, t_X : MX \to X$ and $\circ_X : MX \times_X MX \to MX$ turning every simplicial set X into the object of objects of an internal category. We think of the n-simplices π in MX as *Moore paths* from $s(\pi)$ to $t(\pi)$.

M can be defined as the polynomial functor associated to $\tau : T_* \to T$. Here the n-simplices in T are zigzags of the form

$$\bullet \xleftarrow{p_1} \bullet \xrightarrow{p_2} \bullet \xrightarrow{p_3} \bullet \xleftarrow{p_4} \bullet \xrightarrow{p_5} \bullet$$

with $p_i \in [n]$. The n-simplices in T_* are elements of T_n together with a choice of vertex, while τ is the obvious projection.

(This is not how M was defined in Van den Berg & Garner, but this is an equivalent description.)

Uniform Kan fibration (after 1st modification)

Uniform Kan fibration (one step in the right direction)

A morphism $f : Y \to X$ is a *uniform Kan fibration* if for any commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "m"'] & MY \arrow[d, "{(t, Mf)}"] \\ B \arrow[r] \arrow[ur, dotted] & Y \times_X MX \end{tikzcd}$$

in which m is a cofibration, there is a chosen filler (as indicated), in such a way that for any map $b : B' \to B$ the chosen fillers in

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "b^* m"'] & A \arrow[r] \arrow[d, "m"'] & MY \arrow[d, "{(t, Mf)}"] \\ B' \arrow[r, "b"'] \arrow[urr, dotted] & B \arrow[r] \arrow[ur, dotted] & Y \times_X MX \end{tikzcd}$$

commute which each other.

Second modification

This definition basically says: given a point $y \in Y_n$ and a Moore path $\pi : x' \to x$ in X_n with $fy = x$, we can find another Moore path (a lift) $\rho : y' \to y$ in Y_n with $f(y') = x'$ and $f(\rho) = \pi$, even when you're already told on a cofibrant sieve $S \subseteq \Delta^n$ what the solution should be.

The previous definition already gives that lifts of identity Moore paths lift to identity Moore paths. But we want more:

Second modification

If $\pi = \pi_1\pi_0$ is a composition of those path and y lies over the target of π, then the lift ρ for π given y coincides with the composition of the lift ρ_1 of π_1 given y and the lift ρ_0 of π_0 given $s(\rho_1)$.

$$\begin{tikzcd}
Y \arrow[d, "f"'] & & \bullet \arrow[r, "\rho_0"] \arrow[d, dotted] & s(\rho_1) \arrow[r, "\rho_1"] \arrow[d, dotted] & y \arrow[d, dotted] \\
X & & \bullet \arrow[r, "\pi_0"'] & \bullet \arrow[r, "\pi_1"'] & \bullet
\end{tikzcd}$$

Third (and final) modification

For achieving our goal, making these two modifications are enough. However, at this point it is very likely we will end up adding a third uniformity condition as well.

We have that cofibrations are closed under composition, so if $m : A \to B$ and $n : B \to C$ are cofibrations, we could also demand that the chosen filler for the composition nm coincides with the one obtained by first taking the chosen lift for m and then the chosen lift for n, as in:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "m"'] & MY \arrow[dd, "{(t,Mf)}"] \\
B \arrow[ur, dotted] \arrow[d, "n"'] & \\
C \arrow[r] \arrow[uur, dotted] & Y \times_X MX.
\end{tikzcd}
$$

Section 3

Results

Goal achieved

With these extra uniformity conditions we have:

- uniform Kan fibrations are closed under Π.
- every Kan fibration can be equipped with the structure of a uniform Kan fibration, classically.
- the structure of being a uniform Kan fibration is local.
- every uniform Kan fibration in our sense is also a uniform Kan fibration in the sense of Gambino-Sattler, but we expect the converse to be unprovable constructively.

Towards an algebraic model structure

The main motivation for our work was to give constructive proofs of:

- the existence of an algebraic model structure on simplicial sets.
- the existence of a model of univalent type theory in simplicial sets.

Currently we have constructive proofs/proof sketches for:

- the existence of a model structure on the simplicial sets, when restricted to those that are uniformly Kan.
- the existence of a model of type theory with $\Pi, \Sigma, \mathbb{N}, 0, 1, +, \times$.

Future work

What remains to be proven (constructively!):

- We can show that universal uniform Kan fibration exist, but we haven't shown they are univalent.
- We haven't shown that universes are uniformly Kan.
- And we haven't shown that there exists an algebraic model structure on the entire category of simplicial sets based on our notion of a uniform Kan fibration.

THANK YOU!