Uniform Kan fibrations in simplicial sets
(jww Eric Faber)

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Warning

This is work in progress and not as thoroughly checked as I would have liked!
Section 1

Motivation
Voevodsky

Constructs a model of univalent type theory in simplicial sets, interpreting the dependent types as Kan fibrations.

He builds on well-known properties of the Kan-Quillen model structure to prove such facts as:

**Π-types**

Let \( f : Y \to X \) be a Kan fibration. If \( f^* : sSets/X \to sSets/Y \) is pull back along \( f \), then its right adjoint \( \Pi_f \) (push forward) preserves Kan fibrations.

This is important for interpreting the Π-types in type theory.
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But how constructive is all of this?

**Theorem (Bezem-Coquand-Parmann)**

The classical result which says that if $A$ and $B$ are Kan, then so is $A^B$, is not constructively valid.
Some constructive results

Theorem (Henry)

One can use the standard definitions of a (trivial) Kan fibration to show \textit{constructively} that there is a model structure on simplicial sets.

However, we cannot prove \textit{constructively} that in this model structure every object is cofibrant (!).

Gambino-Henry show how this can be extended to a model of homotopy type theory as well, modulo some issues:

- They only have a weak form of $\Pi$ (constructively).
- An appropriate coherence theorem to turn this into a genuine model of type theory is (so far) missing.

I expect we will hear more about this in the next talk!
Our aim

Define a fibration structure (that of a “uniform Kan fibration”) in simplicial sets such that . . .

- maps with this structure are closed under $\Pi$, constructively.
- every Kan fibration can be equipped with the structure of a uniform Kan fibration, classically.
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Objection

Wait, wasn’t that already done by Gambino & Sattler in their paper “The Frobenius condition, right properness, and uniform fibrations”?
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Response

True, but they were unable to show constructively that their notion of uniform Kan fibration was local (the fibration structure is locally representable).
Local class

Definition

Let us say that a structure on morphisms is \textit{local} if: to equip a morphism \( f : Y \to X \) with this structure it is necessary and sufficient to do this for every pullback of \( f \) along a map \( x : \Delta^n \to X \), in such a way our choices are stable under pulling back along maps \( \alpha : \Delta^m \to \Delta^n \).

\[
\begin{array}{c}
Y_{x \cdot \alpha} \\ \downarrow \\
\Delta^m \\
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow \\
\alpha \\
\end{array} \quad \begin{array}{c}
Y_x \\ \downarrow \\
\Delta^n \\
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow f \\
Y \\
\end{array} \\
\begin{array}{c}
\Delta^m \\
\downarrow \alpha \\
x \\
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow x \\
\Delta^n \\
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow f \\
X \\
\end{array}
\]

Because they were unable to show that their notion of a uniform Kan fibration is local, Gambino & Sattler were unable to show \textit{constructively} that universal uniform Kan fibrations exist.
Our aim, again

Our aim

Define a notion of a uniform Kan fibration in simplicial sets such that . . .

- they are closed under $\Pi$, constructively.
- every Kan fibration can be equipped with the structure of a uniform Kan fibration, classically.
- the structure of being a uniform Kan fibration is local.

Today, I will explain our definition as a modification of the one by Gambino & Sattler. So let's first recall that one.
Section 2

The definition
Cofibrations

Cofibrations (for us)
A map $f : Y \to X$ of simplicial sets is a cofibration if each $f_n : X_n \to Y_n$ is a complemented monomorphism.

Warning
These are not the cofibrations in the Henry model structure!

Note that cofibrations are closed under composition and stable under pullback.
A morphism $f : Y \to X$ is a *uniform Kan fibration* if for any commutative square

\[
\begin{array}{ccc}
A & \rightarrow & Y^I \\
\downarrow m & & \downarrow (s/t,f^I) \\
B & \rightarrow & Y \times_X X^I
\end{array}
\]

in which $m$ is a cofibration, there is a chosen filler (as indicated), in such a way that for any map $b : B' \to B$ the chosen fillers in

\[
\begin{array}{ccc}
A' & \rightarrow & A & \rightarrow & Y^I \\
\downarrow b^* m & & \downarrow m & & \downarrow (s/t,f^I) \\
B' & \rightarrow & B & \rightarrow & Y \times_X X^I
\end{array}
\]

commute which each other. (Here $I = \Delta^1$.)
Simplicial Moore path object

In Van den Berg & Garner, we defined a simplicial Moore path functor. The idea is that there is an endofunctor $M$ on simplicial sets together with natural transformations $r_X : X \to MX$, $s_X$, $t_X : MX \to X$ and $\circ_X : MX \times_X MX \to MX$ turning every simplicial set $X$ into the object of objects of an internal category. We think of the $n$-simplices $\pi$ in $MX$ as Moore paths from $s(\pi)$ to $t(\pi)$.

$M$ can be defined as the polynomial functor associated to $\tau : T_* \to T$. Here the $n$-simplices in $T$ are zigzags of the form

$$
\bullet \leftarrow p_1 \bullet \rightarrow p_2 \bullet \rightarrow p_3 \bullet \leftarrow p_4 \bullet \rightarrow p_5 \bullet
$$

with $p_i \in [n]$. The $n$-simplices in $T_*$ are elements of $T_n$ together with a choice of vertex, while $\tau$ is the obvious projection.

(This is not how $M$ was defined in Van den Berg & Garner, but this is an equivalent description.)
A morphism \( f : Y \to X \) is a uniform Kan fibration if for any commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{m} & MY \\
\downarrow & & \downarrow (t,Mf) \\
B & \xrightarrow{(t,Mf)} & Y \times_X MX
\end{array}
\]

in which \( m \) is a cofibration, there is a chosen filler (as indicated), in such a way that for any map \( b : B' \to B \) the chosen fillers in

\[
\begin{array}{ccc}
A' & \xrightarrow{b^* m} & A & \xrightarrow{m} & MY \\
\downarrow b^* m & & \downarrow m & & \downarrow (t,Mf) \\
B' & \xrightarrow{b} & B & \xrightarrow{(t,Mf)} & Y \times_X MX
\end{array}
\]

commute which each other.
Second modification

This definition basically says: given a point \( y \in Y_{n} \) and a Moore path \( \pi : x' \to x \) in \( X_{n} \) with \( fy = x \), we can find another Moore path (a lift) \( \rho : y' \to y \) in \( Y_{n} \) with \( f(y') = x' \) and \( f(\rho) = \pi \), even when you’re already told on a cofibrant sieve \( S \subseteq \Delta^{n} \) what the solution should be.

The previous definition already gives that lifts of identity Moore paths lift to identity Moore paths. But we want more:

Second modification

If \( \pi = \pi_{1}\pi_{0} \) is a composition of those path and \( y \) lies over the target of \( \pi \), then the lift \( \rho \) for \( \pi \) given \( y \) coincides with the composition of the lift \( \rho_{1} \) of \( \pi_{1} \) given \( y \) and the lift \( \rho_{0} \) of \( \pi_{0} \) given \( s(\rho_{1}) \).

\[
\begin{array}{ccc}
Y & \xrightarrow{\rho_{0}} & s(\rho_{1}) \xrightarrow{\rho_{1}} Y \\
\downarrow f & & \downarrow f \\
X & \xrightarrow{\pi_{0}} & X \\
\end{array}
\]
Third (and final) modification

For achieving our goal, making these two modifications are enough. However, at this point it is very likely we will end up adding a third uniformity condition as well.

We have that cofibrations are closed under composition, so if \( m : A \to B \) and \( n : B \to C \) are cofibrations, we could also demand that the chosen filler for the composition \( nm \) coincides with the one obtained by first taking the chosen lift for \( m \) and then the chosen lift for \( n \), as in:

\[
\begin{array}{ccc}
A & \longrightarrow & MY \\
\downarrow m & & \downarrow (t,Mf) \\
B & \downarrow n & \longrightarrow \\
\downarrow & & \downarrow \\
C & \longrightarrow & Y \times_X MX.
\end{array}
\]
Section 3

Results
Goal achieved

With these extra uniformity conditions we have:

- uniform Kan fibrations are closed under $\Pi$.
- every Kan fibration can be equipped with the structure of a uniform Kan fibration, classically.
- the structure of being a uniform Kan fibration is local.
- every uniform Kan fibration in our sense is also a uniform Kan fibration in the sense of Gambino-Sattler, but we expect the converse to be unprovable constructively.
Towards an algebraic model structure

The main motivation for our work was to give constructive proofs of:

- the existence of an algebraic model structure on simplicial sets.
- the existence of a model of univalent type theory in simplicial sets.

Currently we have constructive proofs/proof sketches for:

- the existence of a model structure on the simplicial sets, when restricted to those that are uniformly Kan.
- the existence of a model of type theory with $\Pi, \Sigma, \mathbb{N}, 0, 1, +, \times$. 
Future work

What remains to be proven (constructively!):

- We can show that universal uniform Kan fibration exist, but we haven’t shown they are univalent.
- We haven’t shown that universes are uniformly Kan.
- And we haven’t shown that there exists an algebraic model structure on the entire category of simplicial sets based on our notion of a uniform Kan fibration.
THANK YOU!