Internal sums for synthetic fibered $(\infty, 1)$-categories*

Jonathan Weinberger[1]

Johns Hopkins University
jweinb20@jhu.edu

Summary: We will present results from [15] about fibrations of $(\infty, 1)$-categories with notions of fibered and extensive coproducts. These generalize 1-categorical results due to Bénabou, Moens, Jibladze, and Streicher [13, 14, 12]. Our framework is Riehl–Shulman's synthetic theory of $(\infty, 1)$-categories [7, 6], working in a simplicial extension of Book HoTT. Thus, qua Shulman's work [10] our results constitute a systematic study of lextensive fibrations of internal $(\infty, 1)$-categories (realized as complete Segal objects [9, 5]) w.r.t. an arbitrary ambient Grothendieck–Rezk–Lurie-$(\infty, 1)$-topos. We anticipate that our work can be used in future applications to higher topos theory [11] as well as homotopy theory [2, 3].

1. From lextensive categories to lextensive fibrations: A category $\mathbf{C}$ is *extensive* if for any small family of objects $(a_i)_{i \in I}$ in $\mathbf{C}$ the coproduct map $\prod_{i \in I} \mathbf{C}/a_i \to \mathbf{C}/\coprod_{i \in I} a_i$ is an equivalence. If $\mathbf{C}$ is a *lex* category, *i.e.*, it all has finite limits, it is called *lextensive*.

In the works of Bénabou, Moens, Jibladze, Streicher, Lietz, and Frey this has been more generally considered in the context of *fibered categories* $p : \mathbf{E} \to \mathbf{B}$ over finitely complete bases $\mathbf{B}$ via the notion of *lextensive* or *Moens fibration*: A fibration $p : \mathbf{E} \to \mathbf{B}$ of finitely complete categories is called *Moens fibration* if it satisfies the following conditions:

1. The fibration p has *internal* or *fibered sums*, *i.e.*, it is also an opfibration and the *Beck–Chevalley condition* is satisfied meaning that both transport operations are compatible in a canonical way.

2. The internal sums of p are *stable*, *i.e.*, cocartesian arrows are closed under pullback along arbitrary maps.

3. The internal sums of p are *disjoint*, *i.e.*, cocartesian arrows are closed under diagonals.

Indeed, in [13, Corollary 15.4] it is shown that given a Moens fibration $p : \mathbf{E} \to \mathbf{B}$ for any $u : i \to j$ and $x \in \mathbf{E}_i$ the functor $\coprod_u /x : \mathbf{E}_i/x \to \mathbf{E}_j/\coprod_u x$ is an equivalence, generalizing the notion of an extensive category to the fibrational setting.

2. Classification via Moens' Theorem: In his doctoral thesis [4] Moens shows that Moens fibrations over a lex base can be characterized up to equivalence as Artin gluings of lex functors whose domain is the base category, *i.e.*, functors of the form $\mathrm{gl}(F) = F^*\mathrm{cod} : \mathbf{C} \downarrow F \to \mathbf{B}$, for a lex functor $F : \mathbf{B} \to \mathbf{C}$ between lex categories and the codomain fibration $\mathrm{cod} : \mathbf{C}^{\to} \to \mathbf{C}$.

3. Lextensive fibrations and Moens' Theorem for $(\infty, 1)$-categories, synthetically: In the classical case of 1-categories lextensive fibrations play an important role in the fibered viewpoint of geometric morphisms due to Bénabou cf. [13, 12]. In light of advancing studies of higher topos theory, particularly as pertaining to the study of models of HoTT [10], we

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believe it to be worthwhile to generalize the mentioned results about extensive fibrations to the setting of $(\infty, 1)$-categories. This is the theme of the work [15] which main results we present here, including a version of Moens' Theorem. We build on our preceding study of synthetic (co)cartesian fibrations joint work with Buchholtz [1] and principles from Riehl–Verity's ∞-cosmos theory [8].

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