Dependent type theories and base change theorems*

Mathieu Anel[1] and Jonathan Weinberger[2]

[1] Department of Philosophy
Carnegie Mellon University
mathieu.anel@protonmail.com
[2] Department of Mathematics
Johns Hopkins University
jweinb20@jhu.edu

The purpose of this work is to advertise some further connections between dependent type theories and geometry/topology. The basic features of type theories are a fibration of dependent types, and the notions of dependents sums and dependent products. We shall present examples of such structures inspired from geometry and topology, potentially providing semantics for type theories. If time allows, we shall even say a word about identity types and universes in those examples.

The general setting will be that of a (∞-)category B of "spatial objects" and a Grothendieck fibration $p : E \to B$ of "coefficients". The existence of dependent sums and products is what algebraic geometers call "base-change theorems". A morphism f in B is called *smooth* (*proper*) if f-dependent sums (products) exist in p. We shall see how these notions recover important classes of maps arising in topology.

1. When $E \to B$ is the fibration of (small) left fibrations over (large) categories, smooth and proper functors recovers Grothendieck's notion of smooth and proper functors.

2. When $E \to B$ is the fibration of open immersions over topological spaces, smooth and proper functors are open and proper morphisms.

3. When $E \to B$ is the fibration of étale maps over 1-topoi, smooth and proper functors are locally connected and tidy geometric morphisms.

4. When $E \to B$ is the fibration of étale maps over ∞-topoi, this recovers locally contractible and proper geometric morphisms in the sense of Lurie.

5. When $E \to B$ is the fibration of parametrized spectra over spaces, dependent sums/products correspond to homology/cohomology, every map is smooth and proper, which is a way to say that the category of spectra is tensored and cotensored over spaces.

6. When $E \to B$ is the fibration of sheaves of spectra over ∞-topoi, dependent sums/products correspond to sheaf homology/cohomology, and they exists respectively along locally contractible/proper geometric morphisms.

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