Eckmann-Hilton and The Hopf Fibration in Homotopy Type Theory

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Both the Eckmann-Hilton argument and the Hopf fibration are familiar constructions to homotopy theorists. Each construction plays an important role in the theory of higher homotopy groups. The Eckmann-Hilton argument is behind the proof that all higher homotopy groups are commutative. It is used to construct a path $\alpha \cdot \beta = \beta \cdot \alpha$, for 2-loops α and β. The Hopf fibration is a map $\mathbb{S}^3 \to \mathbb{S}^2$ that lends a generator of $\Omega^3(\mathbb{S}^2)$, a term that we will call the Hopf 3-loop. A deep connection between the two constructions has been suggested. In a 2011 blog comment, Michael Shulman conjectured that the Eckmann-Hilton argument can be used to construct the Hopf fibration in HoTT. The HoTT book reiterates this claim in the introduction of chapter 8, stating that "the generating element of $\pi_3(\mathbb{S}^2)$ is constructed using the interchange law of [the Eckmann-Hilton argument]". But no proof of this has actually been written in HoTT (at least, not to the author's knowledge). Progress has recently been made in this area by Kristina Sojakova and G. A. Kavvos, who contsruct syllepsis in homotopy type theory, in their 2022 paper of the same name. In this talk I aim to fill the gap in the results by providing a positive answer Shulman's original conjecture. I will present a pen and paper proof in book HoTT that shows that the Hopf 3-loop can be constructed (up to sign) using the Eckmann-Hilton argument. As an immediate application, I will show that this result enables us to calculate $\pi_4(\mathbb{S}^3)$ with syllepsis.

To do this we first review the Eckmann-Hilton argument, which constructs a path $\mathsf{EH}(\alpha, \beta) : \alpha \cdot \beta = \beta \cdot \alpha$, for any 2-loops α and β. We will use this to construct what I will call "the Eckmann-Hilton 3-loop" in $\mathbb{S}^2$. Applying EH to the generating path surf_2 of $\mathbb{S}^2$ and its inverse surf_2^{-1} lends $\mathsf{EH}(\mathsf{surf}_2, \mathsf{surf}_2^{-1}) : \mathsf{surf}_2 \cdot \mathsf{surf}_2^{-1} = \mathsf{surf}_2^{-1} \cdot \mathsf{surf}_2$. By "tying off the ends" with inverse laws, we obtain the Eckmann-Hilton 3-loop $\mathsf{eh} : \Omega^3(\mathbb{S}^2)$.

Then we can use the suspension loop space adjunction to directly construct a map $\mathsf{hpf} : \mathbb{S}^3 \to \mathbb{S}^2$ with $\mathsf{hpf}(\mathsf{surf}_3) = \mathsf{eh}$. We claim that this map is (up to sign) the Hopf fibration. We will show that the fiber of hpf is indeed $\mathbb{S}^1$, which implies that eh is a generator of $\Omega^3(\mathbb{S}^2)$. Thus it must be the same (up to sign) as the Hopf 3-loop. This implies the desired claim.

The bulk of the proof constists in showing that the fiber of hpf is equivalent to $\mathbb{S}^1$. Instead of trying to directly construct an equivalence between the fiber and $\mathbb{S}^1$, we will construct a family $H : \mathbb{S}^2 \to U$ that is $\mathbb{S}^1$ on the base point and then give a fiberwise equivalence $\prod_{x:\mathbb{S}^2} Hx \simeq \mathsf{fib}_{\mathsf{hpf}}(x)$. This is due to a famaliar phenomena in HoTT (present in, e.g., the proof that $\Omega^1(\mathbb{S}^1) \simeq \mathbb{Z}$): we actually need to generalize our claim in order to apply the necessary induction principles. The most interesting parts of the proof are choosing our type family H and constructing the map $\prod_{x:\mathbb{S}^2} H(x) \to \mathsf{fib}_{\mathsf{hpf}}(x)$.

The key insight of the proof, and the motivation for the construction of the family H, comes from some preliminary analysis of the type family $\mathsf{fib}_{\mathsf{hpf}}$. A type family over $\mathbb{S}^2$ is uniquely determined by its descent data. This consists of a type X, which corresponds to the type over the base point of $\mathbb{S}^2$, and a homotopy $\mathsf{id}_X \sim \mathsf{id}_X$, which corresponds to the two dimensional transport $\mathsf{tr}^2(\mathsf{surf}_2) : \mathsf{id} \sim \mathsf{id}$. We can compute this data for $\mathsf{fib}_{\mathsf{hpf}}$ fairly easily as $\mathsf{fib}_{\mathsf{hpf}}(\mathsf{N}_2)$ and

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$\lambda(z,p).(1_z, \mathsf{r\text{-}unit}_p^{-1} \cdot (1_p \star \mathsf{surf}_2) \cdot \mathsf{r\text{-}unit}_p)$. Call this homotopy $(\mathsf{fib})^2$. Since $(\mathsf{fib})^2$ is a homotopy, it comes with an induced naturality condition. We will show that the naturality square induced by the path $(1_{\mathsf{N}_3}, \mathsf{surf}_2^{-1})$ is (almost exactly) the Eckmann-Hilton path.

Now we are in a position to start to choose our type family H. We already know we need the type over the base point to be $\mathbb{S}^1$. But we should expect that $(\mathsf{fib})^2$ is a non-trivial homotopy, since surf_2 is not in the image of hpf. So now we know we need H to have non-trivial 2-dimensional descent data, i.e., a non-trival homotopy $\mathsf{id}_{\mathbb{S}^1} \sim \mathsf{id}_{\mathbb{S}^1}$.

But, what ever homotopy we select, call it $(H)^2$, it has an induced naturality condition. Since $\mathbb{S}^1$ is a 1-type, the naturality condition is essentially trivial (since the naturality cells are 2-paths). Thus, if we are going to have a fiberwise equivalence between H and $(\mathsf{fib})^2$, the naturality condition of $(\mathsf{fib})^2$ must also be trivial. But we have already seen that the naturality of $(\mathsf{fib})^2$ is (more or less) the Eckmann-Hilton path. Thus the proof hinges upon whether or not the Eckmann-Hilton path in the fiber is trivial.

In more detail, the construction of the map $g : \prod_{x:\mathbb{S}^1} Hx \to \mathsf{fib}_{\mathsf{hpf}}(x)$ by $\mathbb{S}^2$-induction will require us to show that our choice of $g(\mathsf{N}_2)$ is natural with respect to the descent data of each type family. This essentially asks for us to construct a homotopy $(\mathsf{fib})^2 \cdot_r g(\mathsf{N}_2) \sim g(\mathsf{N}_2) \cdot_l (H)^2$. We will see that we are practically forced to pick L^{-1} for $(H)^2$, i.e., the homotopy that sends the base point of $\mathbb{S}^1$ to the inverse of the generating loop of $\mathbb{S}^1$. Then, through a few computations, proving this naturality boils down to showing that $(1^2_{\mathsf{N}_3}, \mathsf{eh}) = (1^2_{\mathsf{N}_3}, 1^3_{\mathsf{N}_2})$ in the fiber. But this path space is equivalent to $\mathsf{fib}_{\mathsf{hpf}}(\mathsf{eh})$. And we know this later type is inhabited since $\mathsf{hpf}(\mathsf{surf}_3) = \mathsf{eh}$.

The rest of the proof is fairly routine, using techniques from the HoTT book. Though this proof maybe more involved than the HoTT book's original construction of Hopf fibration, this new proof is a worthwhile endeavor for a few reasons. First, this proof validates an interesting perspective on the complexity of $\mathbb{S}^2$. We can see $\mathbb{S}^2$ as freely generated by a 2-loop. But a 2-loop necessairly comes with "a braiding" that witnesses the commutivity of concatenation of paths (i.e., EH). But this braiding is itself freely generated. Thus, the loop that the braiding induces (i.e., eh) must be a free 3-loop. So, it should generate $\Omega^3(\mathbb{S}^2)$. But then the higher structure (at level 3 and above) of $\mathbb{S}^2$ should be the same as $\mathbb{S}^3$, since they are both freely generated by a 3-loop. This is exactly what an analysis of the fiber sequence of hpf shows us.

Second, this proof provides new and succinct way to calculate $\pi_4(\mathbb{S}^3)$ in HoTT. A quick lemma shows that functions preserve Eckmann-Hilton. That is, $f : \mathbb{S}^2 \to X$ sends eh to the Eckmann-Hilton 3-loop induced by $f(\mathsf{surf}_2)$. Now, $\mathsf{surf}_3 : \Omega^3(\mathbb{S}^3)$ is a 2-loop in $\Omega(\mathbb{S}^3)$, so it determines a map $\mathbb{S}^2 \to \Omega(\mathbb{S}^3)$. But this map turns out to be the unit of the suspension loopspace adjunction. As such, it is 2-connected and, in particular, the induced map $\pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^3)$ is surjective. Thus it sends the generator of $\pi_3(\mathbb{S}^2)$ to the generator of $\pi_4(\mathbb{S}^3)$. But our result establishes that $\pi_3(\mathbb{S}^2)$ is $\mathbb{Z}$ generated by eh. Since functions preserve Eckmann-Hilton, we know that the image of eh under this map, and so the generator of $\pi_4(\mathbb{S}^3)$, is the Eckmann-Hilton 3-loop induced by surf_3. Now we apply Syllepsis to calculate that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.