Internal higher topos theory

Louis Martini[1*] and Sebastian Wolf[2]

[1] Department of Mathematical Sciences, Norwegian University of Science and Technology
louis.o.martini@ntnu.no
[2] Fakultät für Mathematik, Universität Regensburg
Sebastian1.Wolf@mathematik.uni-regensburg.de

One of the main ideas behind the notion of a topos is that it provides semantics for a very rich logic, which can be used to interpret many of the constructions from classical mathematics internal to a topos. In particular, one can interpret the theory of categories itself internally, and even the theory of topoi, which leads to the notion of a topos *internal* to another topos. By a theorem of Moens [Moe82], the datum of such a topos internal to some base topos $\mathcal{B}$ is equivalent to that of a topos $\mathcal{X}$ together with a structure map $\mathcal{X} \to \mathcal{B}$. In other words, the category of topoi internal to $\mathcal{B}$ is equivalent to the category of topoi *over* $\mathcal{B}$.

In my talk, I will report on ongoing joint work with Sebastian Wolf, in which we tell the analogous story in the world of ∞-*topoi*: given an arbitrary base ∞-topos $\mathcal{B}$, we developed a theory of categories internal to $\mathcal{B}$. Such an internal category, hereafter referred to as a $\mathcal{B}$-*category*, is by definition a univalent Segal object in $\mathcal{B}$, i.e. a functor $\Delta^{\mathrm{op}} \to \mathcal{B}$ satisfying certain conditions. Equivalently, a $\mathcal{B}$-category can be encoded by a limit-preserving functor $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$. The theory of $\mathcal{B}$-categories, as developed in our paper series [Mar21, MW21, Mar22, MW22], moves along similar lines as the theory of ∞-categories, so that one can work with $\mathcal{B}$-categories in essentially the same way as with usual ∞-categories:

1. The ∞-category $\mathrm{Cat}(\mathcal{B})$ of $\mathcal{B}$-categories is cartesian closed, so that one obtains a functor $\mathcal{B}$-category $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ for everz pair of $\mathcal{B}$-categories C and D.

2. The ∞-category $\mathrm{Cat}(\mathcal{B})$ of $\mathcal{B}$-categories admits an $(\infty, 2)$-categorical enhancement, which gives rise to a notion of *adjunctions* between $\mathcal{B}$-categories, and in particular of *Bousfield localisations* (which are adjunctions in which the counit is an equivalence).

3. By combining the first two constructions, one obtains a notion of limits and colimits in a $\mathcal{B}$-category C indexed by a $\mathcal{B}$-category I: one simply defines the limit (colimit) functor $\lim_{\mathsf{I}}$ ($\mathrm{colim}_{\mathsf{I}}$) as the right (left) adjoint of the diagonal map $\mathsf{C} \to \underline{\mathrm{Fun}}(\mathsf{I}, \mathsf{C})$.

4. The internal analogue of the ∞-category of spaces $\mathcal{S}$ is played by the *universe* Ω, defined via the limit-preserving functor $\mathcal{B}_{/-} \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$.

5. By combining (1) and (4), one can define the $\mathcal{B}$-category of *presheaves* $\mathsf{PSh}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \Omega)$ on an arbitrary $\mathcal{B}$-category C.

By making use of the building blocks of $\mathcal{B}$-category theory as outlined above, one can now define the notion of a $\mathcal{B}$-*topos*:

Definition. A $\mathcal{B}$-category X is a $\mathcal{B}$-topos if there is a (small) $\mathcal{B}$-category C such that X arises as a left exact and accessible Bousfield localisation of $\mathsf{PSh}(\mathsf{C})$.

The definition of $\mathcal{B}$-topoi is entirely analogous to that of ∞-topoi, and one can similarly prove internal analogues of most statements from higher topos theory, so that $\mathcal{B}$-topoi behave no different than ordinary ∞-topoi. However, from an external point of view, one is now working with ∞-topoi *over* $\mathcal{B}$:

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[*] Speaker.

Theorem. *The ∞-category of $\mathcal{B}$-topoi is equivalent to the ∞-category of ∞-topoi over $\mathcal{B}$.*

For this reason, the language of $\mathcal{B}$-topoi is well-suited for dealing with relative problems in higher topos theory. If time permits, I will discuss some examples where this principle is put to use.

References

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