Finite colimits in an elementary higher topos

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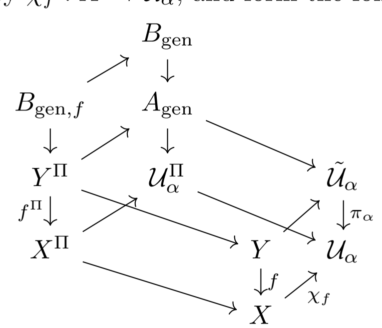
Categorical logic employs concepts from category theory to study mathematical logic, and in particular the relation between type theories and structured categories, such as elementary topoi. The recent rise of higher categories and homotopy type theories suggests an appropriate notion of elementary higher topos, as proposed in [Ras18], whose internal language is given by homotopy type theories.

It is a classical result of 1-topos theory that finite cocompleteness of an elementary topos $\mathbf{E}$ can be deduced from the other axioms. The general idea is that the powerfunctor embedding $P : x \mapsto \Omega^x$ allows one to reverse the direction of arrows when reasoning about a diagram valued in $\mathbf{E}$. More precisely, it can be shown to be part of a monadic adjunction, so that $\mathbf{E}^{op}$ can be identified with the finitely complete category of algebras (see [MM12]).

A similar statement for elementary higher topoi has been investigated, for instance in [FR21], but the pattern of the 1-categorical proof fails to apply in the higher setting for two main reasons. First, the powerfunctor cannot be monadic in general (as it is not even necessarily conservative), accounting for the fact that the mere propositions as defined in HoTT do not generally capture the whole data of the higher structures relevant in intensional type theory. But also, while the usual Beck monadicity theorem only involves colimits of finite diagrams, its higher counterpart (as developed in [RV16]) relies on geometric realizations (that is colimits of simplicial diagrams) which can be thought as the appropriate replacements for coequalizers (notably those corresponding to the quotients of equivalence relations). Geometric realizations being infinite colimits, their existence in the opposite category of an elementary higher topos $\mathcal{E}$ does not follow from finite completeness of $\mathcal{E}$.

The present work, which is currently still in progress, aims at finding a way around those two main hindrances, in order to propose a higher version of the 1-categorical cocompleteness argument in the same spirit.

Let $\mathcal{C}$ be a locally cartesian closed quasicategory with finite limits, enough universes (in the usual sense) and a natural number object $\mathbb{N}$. Consider a universe $\mathcal{U}_\alpha$ and an edge $f : Y \to X$ classified by $\chi_f : X \to \mathcal{U}_\alpha$, and form the following diagram, where all squares are pullbacks:



$B_{\text{gen}} \to A_{\text{gen}} \to \mathcal{U}_\alpha^\Pi$ being the generic composable pair of arrows classified by π_α, as constructed in [KL12]

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We define a functor $P_\alpha : (\mathcal{C}^{\rightarrow}_\alpha)^{op} \to \mathcal{C}^{\rightarrow}_\alpha$, where $\mathcal{C}^{\rightarrow}_\alpha$ is the quasicategory of $\mathcal{U}_\alpha$-small maps in $\mathcal{C}$, by mapping f to $P_\alpha f : f^{\Pi}_* B_{\text{gen},f} \to X^{\Pi}$ as defined in the previous diagram.

P_α can be checked to be adjoint to itself on the right, in the same sense as the powerfunctor in the case of a 1-categorical topos. P_α is also faithful and conservative.

Although we do not have geometric realizations in general, such infinite colimits can be approximated in $\mathcal{C}^{\rightarrow}_\alpha$ and $(\mathcal{C}^{\rightarrow}_\alpha)^{op}$ by what we call internal realizations. An internal realization for a simplicial diagram $X : \Delta^{op} \to \mathcal{C}$ is the internal colimit (as defined in [Ras21]) of any sequence $s : \mathbb{N} \to \mathcal{U}_\alpha$ such that $s \circ n : 1 \to \mathcal{U}_\alpha$ classifies the colimit of the finite diagram $X_n : \Delta^{op}_{\leq n} \to \mathcal{C}$ for every (external) natural number n. If X admits a splitting $X_s : \Delta_\infty \to \mathcal{C}$, the trivial internal realization (that of the constant sequence) coincides with the usual geometric realization. More generally, if $u : \mathcal{D} \to \mathcal{C}$ is faithful, an internal realization of any u-split simplicial diagram $X : \Delta^{op} \to \mathcal{D}$ defines a geometric realization as soon as the corresponding sequence is mapped by u to a trivial one.

Writing $\mathcal{A}_{\mathcal{C},\alpha}$ for the quasicategory of algebras for the monad induced by the previous self-adjunction, any diagram in the family $\mathcal{A}_{\mathcal{C},\alpha} \to ((\mathcal{C}^{\rightarrow}_\alpha)^{op})^{\Delta^{op}}$ should admit an adequate internal realization, as one can construct an associated sequence whose definition barely relies on the existence of finite colimits in $(\mathcal{C}^{\rightarrow}_\alpha)^{op}$.

This allows us to check that the assumptions for the higher Beck monadicity theorem hold, and conclude for an equivalence $\mathcal{A}_{\mathcal{C},\alpha} \simeq (\mathcal{C}^{\rightarrow}_\alpha)^{op}$. The arrow quasicategory $(\mathcal{C}^{\rightarrow})^{op}$, which can be recovered from the family of $(\mathcal{C}^{\rightarrow}_\alpha)^{op}$, is therefore seen to be finitely complete since every $\mathcal{A}_{\mathcal{C},\alpha}$ is so.

References

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