Groupoidal Realizability Over an Untyped Cubical λ-Calculus*

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Realizability is a means to make precise the computational content of a theory. For instance, the original "number realizability" of Kleene [Kle45] associates to each formula in the language of arithmetic a set of natural numbers (the realizers) encoding partial recursive functions that compute witnesses to the truth of the corresponding formula. In this way, realizability is often said to formalize the BHK interpretation. Realizability *categories* are universes of computable mathematics. The notion of computation is usually provided by a partial combinatory algebra (PCA, algebraic model of untyped computation). The most famous example of a realizability category, Hyland's "effective topos" **Eff** [Hyl82], is built over "Kleene's first algebra" $\mathcal{K}_1$ of natural numbers, where $e \cdot n := \phi_e(n)$ is the application of the e^{th} partial recursive function ϕ_e to n. Consequently, a sentence of first-order arithmetic is true in **Eff** iff it is realized in number realizability.

One way to cash out the claim that realizability categories are *universes* of computable mathematics is to say that they model rich and expressive type theories. A significant amount of interest in them stems from their provision of *impredicative* universes, such as that found in the *Calculus of Constructions* [CH88]. Beyond realizability toposes, there are a range of realizability categories: categories of assemblies, partitioned assemblies, and modest subcategories thereof. The category $\mathbf{Asm}(\mathcal{A})$ of assemblies over a PCA $\mathcal{A}$ is regular and locally cartesian closed (so a model of extensional dependent type theory). It has an impredicative universe given by the modest assemblies (equivalently, partial equivalence relations (PERs)) over $\mathcal{A}$. The category $\mathbf{PAsm}(\mathcal{A})$ of *partitioned* assemblies over $\mathcal{A}$ is finitely complete and *weakly* locally cartesian closed, and thus models type theory without function extensionality. There is again an impredicative universe given by the modest partitioned assemblies. Realizability toposes and categories of (partitioned) assemblies are related by free completions: the realizability topos $\mathbf{RT}(\mathcal{A})$ is the ex/lex completion of $\mathbf{PAsm}(\mathcal{A})$ [RR90], and $\mathbf{Asm}(\mathcal{A})$ is the reg/lex completion of $\mathbf{PAsm}(\mathcal{A})$.

There has already been some work on realizability in the context of HoTT. Perhaps the most well-studied approach is that of "cubical assemblies". Uemura [Uem18] has shown that the category of cubical objects in $\mathbf{Asm}(\mathcal{K}_1)$ contains an impredicative and univalent universe that does not satisfy propositional resizing. In other work, van den Berg [vdB18] exhibits **Eff** as the homotopy category of a certain path category in which there is an impredicative and univalent universe of propositions that *does* satisfy propositional resizing. Hofstra and Warren [HW13] equips the syntax of 1-truncated intensional type theory with a notion of realizability, allowing them to show that the syntactic groupoid associated to the type theory generated by a graph has the same homotopy type as the free groupoid on said graph.

But as well as allowing for higher-dimensional structure in the *carrier* objects of realizability models of HoTT, it seems appropriate to allow the *realizers themselves* to carry higher-dimensional structure. This aligns with the extension of the BHK interpretation to higher

*The author would like to thank Samson Abramsky, Michael Warren for some early encouragement, members of the CMU HoTT seminar (especially Steve Awodey for the invitation to speak there and Robert Harper for drawing my attention to his work on higher-dimensional meaning explanations), and Mike Shulman for illuminating discussions about higher-dimensional object classifiers.

dimensions: evidence for an identification is a path (cf. Anguili and Harper [AH17] on higher-dimensional meaning explanations).

We will first describe recent work [Speng] on groupoidal realizability over "realizer categories". These realizer categories are weakly cartesian closed and contain an interval *qua* internal co-groupoid. This facilitates a definition of homotopy inside the category, as well as a fundamental groupoid construction for objects of the category. Objects of a groupoid are realized by points in some fundamental groupoid, and isomorphisms in that groupoid are realized by paths in that fundamental groupoid.

For reasons to do with modelling identity types (that we shall explain), it seems easier to approach groupoidal realizability via *partitioned* assemblies, rather than assemblies in general. We will describe the category $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I})$ of partitioned groupoidal assemblies over the realizer category $\mathbb{C}$ with interval $\mathbb{I}$, and how it gives rise to a model of 1-truncated intensional type theory without function extensionality. If the realizer category $\mathbb{C}$ contains a "universal object" U (every object in $\mathbb{C}$ is a retract of U), the notion of realizability provided by $\mathbb{C}$ can be turned from a *typed* one into an *untyped* one. This allows an investigation of impredicative universes. We will outline the construction of an impredicative universe in $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ given by the "modest fibrations". It is ongoing work to investigate the regular and exact completions of $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$, and possible impredicative and univalent universes there (as well as propositional resizing). There are two settings appropriate for $\mathbf{PGAsm}(\mathbb{C}, \mathbb{I}, U)$ in which free completions have been studied: Shulman has formulated the regular and exact completion of a finitely complete 2-category [Shu21]; van den Berg and Moerdijk have formulated the exact completion of a path category [vM18].

The main topic of this talk, however, will be another line of work in progress. Just as the untyped λ-calculus is an important example of a PCA, we believe that the untyped *cubical* λ-calculus is important for groupoidal realizability. Of course, cubical type theory [CCHM18] is known to provide a computational interpretation of HoTT. Though we say "the" untyped cubical λ-calculus, there are of course various such calculi. We are particularly inspired by the "cubical programming language" of Anguili, Harper and Wilson [AHW17]. In this paper, and later papers in the ensuing series [AFH18, CH19], a PER construction is used to obtain a model of type theory (for increasingly expressive type theories) from a cubical programming language, though *impredicative* universes are not treated.

The untyped λ-calculus was organised by Scott [Sco80] (using the Karoubi envelope, or idempotent completion) into a CCC with universal object. Thus far we have not been able to organise the untyped cubical λ-calculus into a realizer category (CCC with universal object and interval). (We expect to have to quotient by homotopy, but this is not the issue.) Perhaps this construction can be carried out, or perhaps realizer categories need to be generalised. That being said, we are still able to do groupoidal realizability *directly* over the untyped cubical λ-calculus. This is possible because we can form a groupoid out of the syntax of the untyped cubical λ-calculus (perhaps it is appropriate to call this the "fundamental groupoid" of the untyped cubical λ-calculus), allowing us to proceed like before with realizer categories.

It is work in progress to show that the category of partitioned groupoidal assemblies over the untyped cubical λ-calculus gives rise to a model of 1-truncated intensional type theory without function extensionality but with an impredicative universe (which we can hope for as the calculus is untyped). We do not (expect to) need all the bells and whistles found in a cubical type theory, just as in the traditional case of the λ-calculus we do not need, eg. η for functions or constructors for products (they can be encoded, but again won't satisfy η)—a simple form of the Kan filling condition seems to be plenty. Ultimately, we would like to do fully weak ∞-groupoidal realizability over the untyped cubical λ-calculus (and other structures).

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