Symmetric Monoidal Smash Products in HoTT

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In his 2016 proof of $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$, Brunerie [Bru16] crucially uses—but never proves—that the smash product is symmetric monoidal. Due to the vast amount of path algebra involved when reasoning about smash products, this has since remained open. While it turns out that smash products are not needed for Brunerie's proof [LM23; BLM22], the problem is still interesting in its own right. Several attempts have been made at salvaging the situation. Floris Van Doorn [Doo18] came very close to a complete proof by considering an argument using closed monoidal categories but left a gap where the path algebra got too technical. Another line of attack by Cavallo and Harper [CH20; Cav21] is the addition of parametricity. This provides a solution at the cost of complicating the type theory. In this talk, we introduce a heuristic for reasoning about functions defined over smash products and use it to give a complete proof of the fact that the smash product is symmetric monoidal. While all key results have been formalised in `Cubical Agda`[1], the argument is in plain Book HoTT.

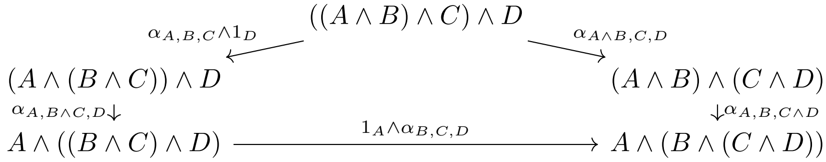
The model of the smash product we will use here is given by the cofibre of the inclusion $A \vee B \hookrightarrow A \times B$. For the sake of clarity, let us spell this out in detail:

Definition 1. *The smash product of two pointed types A and B is the HIT generated by*

- *a point* $\star_\wedge : A \wedge B$
- *for every pair* $(a, b) : A \times B$, *a point* $\langle a, b\rangle : A \wedge B$
- *for every point* $a : A$, *a path* $\mathsf{push_l}(a) : \langle a, \star_B\rangle = \star_\wedge$
- *for every point* $b : B$, *a path* $\mathsf{push_r}(b) : \langle \star_a, b\rangle = \star_\wedge$
- *a coherence* $\mathsf{push_{lr}} : \mathsf{push_l}(\star_A) = \mathsf{push_r}(\star_B)$.

The fact that the smash product is commutative is very direct. Its associativity, however, is harder to prove. This was first proved, in HoTT, by van Doorn [Doo18], using the adjunction $(A \wedge B \to_\star C) \simeq (A \to_\star (B \to_\star C))$ and by Brunerie [Bru18], using a computer generated proof in `Agda`. Here, we give an explicit proof by considering a more involved HIT $\bigwedge(A, B, C)$ satisfying $(A \wedge B) \wedge C \simeq \bigwedge(A, B, C)$ and (trivially) $\bigwedge(A, B, C) \simeq \bigwedge(C, A, B)$. This automatically gives the desired equivalence $\alpha_{A,B,C} : (A \wedge B) \wedge C \simeq A \wedge (B \wedge C)$. The advantage of this explicit description of the equivalence is that it becomes easier to trace. In particular, it is easy to understand its behaviour when applied to homogeneous points and 1-dimensional path constructors. This will turn out to be precisely what we need.

The key problem in proving the fact that the smash product is symmetric monoidal is verifying MacLane's pengaton, i.e. the (pointed) commutativity of the following diagram.



[1] The formalisation is available at https://github.com/aljungstrom/cubical/blob/pentagon/Cubical/HITs/SmashProduct/SymmetricMonoidal.agda (L562). Note that the definition of *precategories* is non-standard.

A naive proof attempt by smash product induction forces us to fill (highly non-trivial) cubes up to 5 dimensions and quickly descends into coherence hell. It turns out that we never need to verify any coherences coming from the $\mathsf{push}_{\mathsf{lr}}$ constructor, but this still is not enough. We need a stronger induction principle. To this end, we recall that a pointed type A is called *homogeneous* if for any point $a : A$ we have an equality of pointed types $(A, \star_A) = (A, a)$. We turn our attention to an incredibly useful lemma, first conjectured for Eilenberg-MacLane spaces in work leading up to [BLM22] and later proved and generalised by Cavallo (and later further generalised by Buchholtz, Christensen, G. Taxerås Flaten and Rijke [Buc+23]) which states that any two pointed functions $f, g : A \to_\star B$ with B homogeneous are equal as pointed functions iff their underlying functions are equal. Using the fact that $A \to_\star B$ is homogeneous whenever B is, the adjunction $(A \wedge B \to_\star C) \simeq (A \to_\star (B \to_\star C))$ yields the following result.

Lemma 1. *Let $f, g : A \wedge B \to_\star C$ with C homogeneous. We have $f = g$ as pointed functions iff $f\langle a, b\rangle = g\langle a, b\rangle$ for $(a, b) : A \times B$.*

If we could apply Lemma 1 to functions $((A \wedge B) \wedge C) \wedge D \to A \wedge (B \wedge (C \wedge D))$, the pentagon would be trivial. Unfortunately, arbitrary smash products are *not* necessarily homogeneous. Fortunately, there is still some use for the lemma. Let us consider the following construction.

Definition 2. *Let $f, g : A \wedge B \to C$ and $h : ((a, b) : A \times B) \to f\langle a, b\rangle = g\langle a, b\rangle$. We define two pointed functions $L_h : A \to_\star f(\star_\wedge) = g(\star_\wedge)$ and $R_h : B \to_\star f(\star_\wedge) = g(\star_\wedge)$ by*

$$L_h(a) = (\mathsf{ap}_f(\mathsf{push}_\mathsf{l}(a)))^{-1} \cdot h(a, \star_B) \cdot \mathsf{ap}_g(\mathsf{push}_\mathsf{l}(b))$$
$$R_h(b) = (\mathsf{ap}_f(\mathsf{push}_\mathsf{r}(b)))^{-1} \cdot h(\star_A, b) \cdot \mathsf{ap}_g(\mathsf{push}_\mathsf{r}(b))$$

where we may simply take $f(\star_\wedge) = g(\star_\wedge)$ to be pointed by either $L_h(\star_A)$ or $R_h(\star_A)$ (these are equal by $\mathsf{push}_{\mathsf{lr}}$, so the choice does not matter).

We can easily derive the following induction principle.

Lemma 2. *Let $f, g : A \wedge B \to C$. The following data gives an equality $f = g$:*

- *A homotopy $h : ((a, b) : A \times B) \to f\langle a, b\rangle = g\langle a, b\rangle$*
- *Equalities of pointed functions $L_h = \mathsf{const}_{L_h(\star_B)}$ and $R_h = \mathsf{const}_{R_r(\star_B)}$.*

The second datum above looks almost absurd: it is asking us to provide equalities of pointed functions, which is much stronger than what is actually needed. However, the codomain of these functions is homogeneous, so we need not worry. In particular, when e.g. A is another smash product, as is the case in the pentagon, Lemma 1 applies which in effect makes this part of the proof trivial. In fact, Lemma 2 and Lemma 1 may be iteratively applied to any n-fold smash product, completely removing the need to verify any higher coherences. Let us state this as an informal theorem:

Theorem 1 (Informal)**.** *To show that two functions $f, g : ((A_1 \wedge A_2) \wedge \dots) \wedge A_n \to B$ are equal, it suffices to provide a family of paths $f\langle x_1, \dots, x_n\rangle = g\langle x_1, \dots, x_n\rangle$ for $x_i : A_i$ and to show that this is coherent with f and g on any **single** application of push_l or push_r.*

This applies to all non-trivial proofs related to the symmetric monoidal structure of the smash product, and in particular to the pentagon. The pentagon holds by definition for homogeneous elements $\langle a, b, c, d\rangle : ((A \wedge B) \wedge C) \wedge D$, so we are only left to trace single instances of the push constructors, which turns out to be very direct (albeit somewhat lengthy). The pointedness requirement is equally direct. Theorem 1 can be used to show the remaining axioms and we easily arrive at the main result:

Theorem 2. *The smash product is symmetric monoidal with the type of booleans as unit.*

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