Cofibrancy of The Exo-type of Natural Numbers

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In this talk, we discuss the axiom that the exo-type of natural numbers (exo-nat), $\mathbb{N}^e$, is cofibrant. We both present what we gain from assuming it and mention its semantics. Also, we briefly present a formalization of the study in Agda.

Two-level type theory (2LTT) [2] combines two type theories: one level as HoTT and the second level as TT validating the uniqueness of identity proofs. Following the literature [1], we call the types in HoTT as usual and those in the other level "exo-types". If A is an exo-type isomorphic to a type B, then A is called *fibrant*. We can weaken this definition. An exo-type A is called *cofibrant* if, for any family of types Y over A, the exo type $\prod^e_{a:A} Y(a)$ is fibrant, and if each $Y(a)$ is contractible, then the fibrant match of $\prod^e_{a:A} Y(a)$ is contractible. We present another but equivalent definition of cofibrancy.

Cofibrancy is preserved under dependent sums and coproducts. It does not seem to be possible to prove that $\mathbb{N}^e$ is cofibrant [1], but it is sometimes added as an axiom (called A3 in [2]). Using this, we showed that cofibrancy is preserved under list types and binary tree types. In [1], it has been proven that if $\mathbb{N}^e$ is cofibrant, it is *sharp*; namely, it has a fibrant replacement. After obtaining new rules about cofibrancy, we tried to generalize the criteria for being cofibrant. At least, any exo-type that can be written as a dependent

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sum of cofibrant exo-types, and(or) $\mathbb{N}^e$ is cofibrant. In particular, a record[1] exo-type of cofibrant exo-types is cofibrant.

We also formalized all these results about cofibrancy in Agda[2]. We used one of the new features of Agda that enable a sort *SSet* for exo-types. One can read the details of this feature in the documentation[3].

It is known that categories with families (CwF) [3] is used to build a model of 2LTT. A CwF has to be enhanced to the appropriate kind of ”two-level CwF” to model 2LTT. For example, as a presheaf category, the category of simplicial sets is one such model. We analyze the semantics of cofibrancy and investigate the models that satisfy cofibrant exo-nat. The previous example is one such model, but it is a trivial consequence. We also analyze the criteria for a Cwf to satisfy cofibrant exo-nat. Our study is still in progress.

Although it is in progress, we try to generalize the cofibrancy rules for general W-exo-types. It is reasonable to conclude that it should be added as an axiom because W exo-types include exo-nat. However, its semantics also should be analyzed.

References

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[3] Peter Dybjer. Internal type theory. In Stefano Berardi and Mario Coppo, editors, Types for Proofs and Programs (TYPES), volume 1158 of Lecture Notes in Computer Science, pages 120–134. Springer-Verlag, 1995

[1] https://agda.readthedocs.io/en/v2.6.3/language/record-types.html

[2] https://github.com/UnivalencePrinciple/2LTT-Agda

[3] https://agda.readthedocs.io/en/v2.6.3/language/two-level.html