Duality for Clans and the Fat Small Object Argument*

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Cartmell's generalized algebraic theories [Car86] — which extend algebra by type dependency — and Freyd's essentially algebraic theories [Fre72] — which permit a controlled form of partiality — are commonly recognized as being equally expressive, and are both subsumed by the less syntactic and more abstract *cartesian theories*, which are simply small finite-limit categories.

More specifically, for each generalized/essentially algebraic theory $\mathcal{T}$ there exists a small finite-limit category $\mathbb{C}$ such that

$$\mathsf{Mod}(\mathcal{T}) \simeq \mathsf{Lex}(\mathbb{C}, \mathsf{Set}),$$

i.e. the category of models of $\mathcal{T}$ is equivalent to the category of finite-limit-preserving ('lex') functors from $\mathbb{C}$ to the category of sets.

Classical Gabriel-Ulmer duality [GU71] states that the categories of models of such theories are precisely the *locally finitely presentable categories*, giving rise to a contravariant biequivalence

$$\mathbf{Lex} \simeq \mathbf{LFP}^{\mathsf{op}} \tag{1}$$

between the 2-categories of finite-limit categories and locally finitely presentable categories.

We present a *refinement* of this duality based on the notion of *clan*, which was introduced by Taylor under the name 'category with display maps' [Tay87, 4.3.2], and later renamed by Joyal [Joy17, 1.1.1].

Definition 1. *A* clan *is a small category $\mathcal{T}$ with terminal object* 1 *equipped with a class $\mathcal{D}$ of* display maps *such that*

1. *pullbacks of display maps along arbitrary maps exist and are again display maps, and*
2. *display maps contain isomorphisms and terminal projections and are closed under composition.*

A model *of a clan $\mathcal{T}$ is a functor $A : \mathcal{T} \to \mathsf{Set}$ which preserves* 1 *and pullbacks of display maps. We denote the full subcategory of $[\mathcal{T}, \mathsf{Set}]$ on models by $\mathsf{Mod}(\mathcal{T})$.*

Since corepresentable functors $\mathcal{T}(\Gamma, -)$ preserve all limits, the Yoneda embedding lifts to a fully faithful functor $Z : \mathcal{T}^{\mathsf{op}} \to \mathsf{Mod}(\mathcal{T})$. Clans refine cartesian theories in that the 'same' finite-limit theory can be represented by different clans, thus we cannot expect to reconstruct a clan from its category of models alone. To recover a duality we equip $\mathsf{Mod}(\mathcal{T})$ with additional information in form of a *weak factorization system* $(\mathcal{E}, \mathcal{F})$ which is cofibrantly generated by the set

$$\mathcal{E}_0 = \{Z(f) \mid f \in \mathcal{D}\},$$

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of morphisms. We call maps in $\mathcal{F}$ *full maps*, maps in $\mathcal{E}$ *extensions*, and models A such that $(0 \to A) \in \mathcal{E}$ 0-*extensions*.

We then obtain a refinement

$$\mathsf{Clan} \simeq \mathsf{ClanAlg}^{\mathsf{op}}$$

of the biequivalence (1), where Clan is the 2-category of clans, and $\mathsf{ClanAlg}$ is the category of *clan algebraic categories*, i.e. locally finitely presentable categories $\mathfrak{A}$ equipped with a weak factorization system $(\mathcal{E}, \mathcal{F})$ such that

1. the full subcategory $\mathbb{C} \subseteq \mathfrak{A}$ on finitely presented 0-extensions is dense,
2. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by $\mathcal{E} \cap \mathrm{mor}(\mathbb{C})$, and
3. $\mathfrak{A}$ has full and effective quotients of component-wise full equivalence relations.

The proof of this result crucially relies on the *fat small object argument* [MRV14], and the last part of the talk is devoted to discussing two variants of this argument, which simplifies considerably in the case of clans.

References

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