Ext groups in homotopy type theory

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We begin the study of homological algebra in homotopy type theory by developing the theory of Ext groups of R-modules. Ext groups are important algebraic invariants, and also have many applications in homotopy theory. For example, the results discussed here were used in [BCFR23] to show that certain types must be products of Eilenberg–Mac Lane spaces. Classically, Ext groups are ingredients in the universal coefficient theorem for cohomology, and we hope to use the results here to obtain a universal coefficient spectral sequence in homotopy type theory. As we will mention below, one cannot assume that there are enough injective or projective modules in HoTT. As a result, we define our Ext groups to be *Yoneda* Ext groups. A good reference for the classical theory of Yoneda Ext groups in the spirit that we present below is [ML63].

We follow the conventions of [Uni13], including the use of higher inductive types and a hierarchy of univalent universes. All of our groups, rings and modules are assumed to be sets.

Let R be a ring and let A and B be R-modules. The type $\mathsf{SES}_R(B, A)$ is defined to be the type of all short exact sequences

$$0 \longrightarrow A \xrightarrow{i_E} E \xrightarrow{p_E} B \longrightarrow 0,$$

where E is any R-module. We define $\mathsf{Ext}^1_R(B, A)$ to be $\pi_0(\mathsf{SES}_R(B, A))$. We often denote a short exact sequence by its middle group E.

It is straightforward to see that $\mathsf{SES}_R(B, A)$ is a 1-type, and that for two short exact sequences E and F, the type $E = F$ is equivalent to the type of R-module isomorphisms $E \cong F$ commuting with the structure maps. From this it follows that $\pi_1(\mathsf{SES}_R(B, A)) \cong \mathsf{Mod}_R(B, A)$ as groups.

We show that Ext^1_R is a contravariant functor in the first variable and a covariant functor in the second variable, under natural notions of pullback and pushforward of extensions. We also show that $\mathsf{Ext}^1_R(-, A)$ sends coproducts to products, while $\mathsf{Ext}^1_R(B, -)$ preserves finite products. It follows from the latter that $\mathsf{Ext}^1_R(B, A)$ is an abelian group. The operation is called the Baer sum.

One aspect of our definition of Ext groups is that they live in a larger universe, since the middle group E runs over a large type. Indeed, we believe that there is an elementary ∞-topos in which the *external* Ext groups over the interpretation of the integers can be proper classes. However, we in fact show that our (internal) Ext groups are always equivalent to small types, and hence are small in models. When $R = \mathbb{Z}$, we do this by proving an equivalence

$$\mathsf{SES}_{\mathbb{Z}}(B, A) \;\simeq\; \big(\mathsf{K}(B, n) \to_* \mathsf{K}(A, n+1)\big)$$

for each $n \geq 2$. It follows that $\mathsf{SES}_{\mathbb{Z}}(B, A)$ is essentially small, and therefore that $\mathsf{Ext}^1_{\mathbb{Z}}(B, A)$ is essentially small. Since the set of R-module structures on an abelian group E is small, it also

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follows that $\mathsf{SES}_R(B,A)$ and $\mathsf{Ext}^1_R(B,A)$ are essentially small. The equivalence is also useful in other ways, for example, in [BCFR23, Theorem 5.13].

The definition of the higher Ext groups is more subtle. For $n \geq 1$, we inductively define

$$\mathsf{ES}^1_R(B,A) :\equiv \mathsf{SES}_R(B,A) \quad \text{and} \quad \mathsf{ES}^{n+1}_R(B,A) :\equiv \sum_{C:\mathsf{Mod}_R} \mathsf{SES}_R(B,C) \times \mathsf{ES}^m_R(C,A).$$

Then, for $n \geq 2$, we define a relation $\sim$ on $\mathsf{ES}^n_R(B,A)$ by

$$(C, E_0, E_1) \sim (D, F_0, F_1) :\equiv \sum_{\beta: C \to C'} (\beta E_0 \sim F_0) \times (E_1 = F_1 \beta),$$

where βE_0 and $F_1\beta$ denote the pushforward and pullback of extensions along β. We let $\mathsf{Ext}^n_R(B,A)$ be the set-quotient $\mathsf{ES}^n_R(B,A)/{\sim}$. Using this definition, one can prove the existence of the usual long exact sequences of Ext groups [Fla23].

It may be surprising that these higher Ext groups can be nonzero in models, even when $R = \mathbb{Z}$. However, we prove that they vanish in common cases, using the following material.

We say that an R-module P is **projective** if for every R-module A, every R-linear epimorphism $A \to P$ merely has an R-linear section. It is equivalent to ask that $\mathsf{Ext}^1(P,A) = 0$ for every R-module A. It follows from the long exact sequence that if an R-module B has a projective resolution, then this resolution can be used to compute $\mathsf{Ext}^n_R(B,A)$ in the usual way. In this situation, it follows that $\mathsf{Ext}^n_R(B,A)$ is essentially small, but we do not know if this is true in general.

The integers are a PID in the constructive sense of [LQ15]. We prove that if R is a PID and B is a finitely presented R-module, then $\mathsf{Ext}^n_R(B,A) = 0$ for $n > 1$. Moreover, if A is also finitely presented, then $\mathsf{Ext}^1(B,A)$ is as well. These results follow from the existence of a length two projective resolution of B using finitely generated free modules.

We also prove results about the case in which $R = \mathbb{Z}G$, the group ring of a group G. For example, we show that if M is a $\mathbb{Z}G$ module, then $H^1(BG; M) \simeq \mathsf{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, M)$, where the left-hand side is the cohomology of BG with local coefficients in M and on the right-hand side, $\mathbb{Z}$ has trivial G-action.

We also study the interpretation of Ext groups in an ∞-topos, making use of [Fla22], and show that they reproduce *sheaf* Ext groups in slices of 1-localic ∞-toposes. We also discuss how these internal Ext groups can behave quite differently from external Ext groups.

Many of our results have been formalized in the Coq-HoTT library [CH].

References

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