Central H-spaces and banded types

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We define and study *central* types A and types *banded by* A, or A*-bands* for short [BCFR23]. Centrality is an elementary condition with perhaps surprising consequences: we will use it to construct an infinite delooping of A and to give concrete formulas for delooping any pointed self-map $A \to_* A$. Moreover, these deloopings will be unique. Examples of central types are given by Eilenberg–Mac Lane spaces and some products of such, for which the mentioned facts are well known. However, even for Eilenberg–Mac Lane spaces, our concrete construction of the deloopings is new, and seems well-suited for certain cohomology computations.

Given a type A, any $a : \|A\|_0$ determines a **path component** $A_{(a)} :\equiv \sum_{x:A} \|a = x\|$ of A. We denote base points by **pt**. A pointed type A is **central** when the *evaluation fibration*

$$f \longmapsto \mathrm{ev}_{\mathrm{id}}(f) :\equiv f(\mathbf{pt}) \;:\; (A \to A)_{(\mathrm{id})} \longrightarrow A,$$

which evaluates a map in the component of id at **pt**, is an equivalence. The terminology comes from higher group theory: if A is connected, then $(A \to A)_{(\mathrm{id})}$ is the delooping of the (higher) center of the ∞-group ΩA. Thus centrality asserts that the "inclusion" (which is just a map in the higher setting) of the center into A is an equivalence.

Let A be a central type. It follows that A is a connected H-space, since the domain of $\mathrm{ev}_{\mathrm{id}}$ is. Moreover, one checks that the loop space of $\mathrm{BAut}_1(A) :\equiv \sum_{X:\mathcal{U}} \|A = X\|_0$, based at $A_{|\mathtt{refl}|_0}$, is equivalent to $(A \to A)_{(\mathrm{id})}$. Thus $\mathrm{BAut}_1(A)$ is a delooping of A under the centrality assumption. It follows from [Rij17] that $\mathrm{BAut}_1(A)$ is equivalent to a small type. The notation comes from $\mathrm{BAut}_1(A)$ being the 1-connected cover of $\mathrm{BAut}(A)$. It consists of types X_p **banded by** $\boldsymbol{A}$, i.e., types X equipped with a **band** $p : \|A = X\|_0$. This concrete description of the delooping of A is what underlies the proofs and results of [BCFR23].

Theorem. *The type* $\mathrm{BAut}_1(A)$ *is the unique delooping of* A*.*

We sketch the proof. Given any delooping B of A, i.e., a pointed, connected type B equipped with a pointed equivalence $\phi : \Omega B \simeq_* A$, we construct an equivalence $f : B \to_* \mathrm{BAut}_1(A)$ of deloopings. For $b : B$, the underlying type of $f(b)$ is $(\mathbf{pt} =_B x)$. Now, since A is connected, B is simply connected. So to get a map $f : B \to \mathrm{BAut}_1(A)$, it suffices to give a band $\|f(\mathbf{pt}) = A\|_0$ of $f(\mathbf{pt})$. Using univalence, ϕ gives us such a band. Theorem 4.6 of loc. cit. has further details.

We also give a concrete and curious formula for delooping pointed self-maps $A \to_* A$. We write $\mathrm{id}^*(a) :\equiv a \backslash \mathbf{pt}$ for the left inversion map on A, where $\backslash$ denotes division.

Theorem. *Every pointed map* $f : A \to_* A$ *has a unique delooping, given by*

$$X_p \longmapsto (X \to A)_{(f \circ \mathrm{id}^* \circ \tilde{p}^{-1})} \;:\; \mathrm{BAut}_1(A) \to_* \mathrm{BAut}_1(A).$$

Here $\tilde{p}^{-1}$ is the inverse equivalence corresponding to the band p, which makes sense for defining the component above. The theorem implies that the loops functor gives an equivalence

$$\Omega : \big(\mathrm{BAut}_1(A) \to_* \mathrm{BAut}_1(A)\big) \xrightarrow{\;\sim\;} (A \to_* A).$$

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We will use this equivalence to show that $\mathrm{BAut}_1(A)$ is central whenever A is, thus letting us iterate to get infinite deloopings of A and of pointed self-maps of A. The key to doing this is the following characterization: *a connected H-space X is central if and only if the type $X \to_* X$ is a set.* Thus, by giving an H-space structure on $\mathrm{BAut}_1(A)$ we deduce its centrality from the equivalence above.

For two A-bands X_p and Y_q, we make $(X_p = Y_q)$ into an A-band by inducting on p and q and using centrality. However, the path types do not give multiplication on $\mathrm{BAut}_1(A)$, but division. To get multiplication, we have to invert the left argument, as follows. The **dual $\boldsymbol{X_p^*}$** of X_p is the type X equipped with the band $|\mathtt{refl}^*|_0 \cdot p$, where $\mathtt{refl}^*$ is the path corresponding to id^*. (Here we used composition of bands.) The tensor product of bands is then defined as

$$X_p \otimes Y_q \;:\equiv\; (X_p^* = Y_q).$$

Tensoring is a symmetric binary operation on $\mathrm{BAut}_1(A)$. Moreover, we show that the symmetry $\sigma_{X_p,Y_q} : (X_p \otimes Y_q) = (Y_q \otimes X_p)$ satisifies $\sigma_{\mathbf{pt},\mathbf{pt}} = \mathtt{refl}$. From this we get an H-space structure by only proving one unit law of $\otimes$ and inducing the other from symmetry. The map $(\mathbf{pt} \otimes X_p) \to X_p$ which transports the base point of A along a path underlies the left unit law.

Theorem. *The operation $\otimes$ makes $\mathrm{BAut}_1(A)$ into a (connected) H-space, hence a central type.*

As a corollary, we have that any central type is an infinite loop space (in a unique way), and similarly for pointed self-maps of a central type.

The notion of A-bands turns out to be equivalent to a natural notion of *A-torsor*. The latter has been independently studied in [Wär23] under the more general condition that A admits a unique H-space structure. In ongoing work, Wärn has given abstract proofs of the theorems above under this weaker assumption. We also expect that many of the formulas we have given have analogues in this more general setting, and are currently investigating this.

It is easy to see that any unique H-space structure $\mu : A^2 \to A$ is symmetric, and from this to define a binary operation $\tilde{\mu} : \sum_{X:\mathrm{BAut}(2)} A^X \to A$ on the type of unordered pairs in A. When A is central, we extend this operation to the type of *genuine unordered pairs* [Buc23].

An open question is whether there exist central types which are not a products of Eilenberg–Mac Lane spaces (GEMs), as all known central types are GEMs. Using results on Ext groups from [Fla23; CF23], we show that any truncated central type with two non-zero, finitely generated homotopy groups is necessarily a GEM.

References

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