Biased elementary doctrines and quotient completions*

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Quotient completions are a common construction in mathematics and logic, extensively studied in category theory. An explicit description of the free exact category of a left exact one was provided in [1]. Later, this result was extended to weakly left exact categories in [2]. In [8], Maietti and Rosolini introduced the elementary quotient completion to provide an abstract description of the quotient construction in [5] for the Minimalist foundation [9]. To achieve this, they relativize the notions of equivalence relation and quotient for Lawvere's elementary doctrines, which are suitable functors of the form $P : \mathcal{C} \to \mathsf{Pos}$, from a category $\mathcal{C}$ with strict finite products to the category Pos of posets and order preserving functions [3, 4]. As shown in [8, 7], the elementary quotient completion generalizes the exact completion of a category with weak finite limits but strict finite products, although in general it does not give rise to an exact category.

In this talk we aim to fill the gap between the elementary quotient completion and the exact completion of weakly left exact categories.

To achieve this, we introduce the notion of *biased* elementary doctrine, which is a suitable functor $P : \mathcal{C}^{op} \to \mathsf{Pos}$ from a category with weak finite products. The structure of biased elementary doctrines is similar to the classical one, but the properties are restated taking into account a sort of *bias* due to the weak universal property of weak finite products.

We present two main constructions for these structures. The first one is the *strictification*, which associates to every biased elementary doctrine $P : \mathcal{C}^{op} \to \mathsf{Pos}$ an elementary doctrine $P^s : {\mathcal{C}_s}^{op} \to \mathsf{Pos}$ on the finite product completion $\mathcal{C}_s$ of $\mathcal{C}$. The second is a quotient completion which extends both the elementary quotient completion and the exact completion of weakly left exact categories, even in case of weak finite products.

Our main example comes from the *intensional level* of the Minimalist foundation [9, 5], which has recently been shown to be compatible with Homotopy type theory [6]. The syntactic category of this theory, denoted by $\mathcal{CM}$, has strict finite products and weak pullbacks and is equipped with an elementary doctrine $G^{\mathbf{mTT}} : \mathcal{CM}^{op} \to \mathsf{Pos}$. For each object $A \in \mathcal{CM}$, the slice category $\mathcal{CM}/A$ inherits a functor, the *slice doctrine*, ${G^{\mathbf{mTT}}}_{/A} : \mathcal{CM}/A \to \mathsf{Pos}$, which has the structure of a biased elementary doctrine.

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