Oracle modalities*

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In this talk I'll introduce a new forumlation of Turing reducibility based on the theory of higher modalities [RSS20] and building on some key ideas due to Hyland [Hyl82, Section 17].

Through cubical assemblies [SU21] we see that it is consistant with HoTT that all functions $\mathbb{N} \to \mathbb{N}$ are computable. However, despite this, cubical assemblies do contain non computable functions: we can include the category of sets in cubical assemblies by first viewing a set as a uniform assembly, and then viewing the assembly as a discrete cubical assembly. We can access these functions from the internal language of cubical assemblies via double negation sheaves. Explicitly, using the results of [Swa22] we can define the modality, ∇, of 0-truncated $\neg\neg$-sheafification in cubical assemblies. We can then think of maps $\chi : \mathbb{N} \to \nabla\mathbb{N}$ as external functions in sets that are not necessarily computable. Given χ we can define an *oracle modality* $\bigcirc_\chi$ as the smallest modality forcing χ to extend to a total function from $\mathbb{N}$ to $\mathbb{N}$, formally defined using *nullification.* Functions $\mathbb{N} \to \mathbb{N}$ in the reflective subuniverse corresponding to $\bigcirc_\chi$ can be thought of as functions that are computable using χ as an *oracle*.

The class of all modalities can be naturally viewed as a preorder by setting $\bigcirc \leq \bigcirc'$ when every $\bigcirc'$-modal type is $\bigcirc$-modal. Applying this ordering to oracle modalities gives us a new way of looking at the preorder of *Turing degrees*, an important structure in computability theory.

I'll discuss applications to the field of *synthetic computability theory* [Ric83, Bau06, Bau17, FJ23] and give some basic examples of oracle modalities interacting with higher types.

References

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