Presheaf Models of Polarized Higher-Order Abstract Syntax

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In recent years, there have been a number of proposals for how to formulate **directed homotopy type theory**[LH11, Nuy15, RS17, Nor19, WL20] – a hypothetical variant of HoTT which replaces identity types with asymmetric *hom types*, providing a language for synthetic category theory. One of the key ingredients for a directed type theory is a robust calculus of *polarity*, allowing for reasoning about co- and contra-variance within the theory. Most styles of directed type theory accomplish this by having a 'negation' operation on types, i.e. positing a negative type A^- for each type A. Following [Alt19, Ses19], we pursue a type theory with a kind of 'deep' polarity: not only do we have negation on types, but also negation on *contexts* and *negative context extension*. This type theory originates from a directed analogue of the groupoid model of type theory[HS95] – the *category model* of directed type theory – but the salient features can be abstracted to a general notion of a **polarized category with families (PCwF)**.[1]

A well-known shortcoming of categories with families [Dyb95] and other similar models of type theory is that they are *first-order*, i.e. they explicitly model the calculus of substitutions and variable bindings of the object theory. Consequently, every type- and term-former of the object theory must be introduced with substitution rules to guarantee stability under substitution (see e.g. [Hof97, 3.3]). For complex systems like dependent type theory, proves quite cumbersome. This shortcoming can be overcome by instead working in a **higher-order abstract syntax (HOAS)**[PE88, HHP93], which encodes variable binding as metatheoretic functions and makes stability under substitution implicit. Moreover, higher-order abstract syntax can be given semantics in presheaf categories[Hof99], whose category-theoretic properties are well-understood.

The goal of this work is to develop a confluence of these two lines of research: higher-order abstract syntax for polarized type theory. One obstacle to overcome is that HOAS does not make explicit mention of contexts, thus making it difficult to represent our desired operation of context negation. However, a modification to our notion of PCwF – inspired by contemplating a polarized variant of *natural models*[Awo18] – allows the construction to go through. We describe the presheaf model of polarized type theory, the construction of Hofmann-Streicher universes[HS99], the interpretation of polarized HOAS into the presheaf model, and the interplay between polarities and dependent types. Time permitting, we'll propose how to extend polarized type theory into directed type theory by the addition of *core types* and hom-types, show what synthetic category theory looks like in this setting, as well as discuss the connection between this work and the burgeoning branch of HoTT known as *higher observational type theory*[Shu22, AKS22].

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