Colimits in the category of pointed types*

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Homotopy type theory is a useful system for developing synthetic homotopy theory. We treat types as spaces and function types as hom-groupoids, thereby making the universe of types an internal version of the ∞-category of spaces. In certain cases, we want to prove things about the category $\mathcal{U}^*$ of *pointed* types and basepoint-preserving functions $A \to_* B :\equiv \sum_{f:A\to B} f(a_0) = b_0$. In our ongoing work on a type-theoretic version of the Brown representability theorem, we need a Yoneda-like lemma whose standard proof requires pointed function types to take colimits to limits. Since we must carry proofs of the identity $f(a_0) = b_0$, not all colimits, defined as 1-HITs in the standard way, satisfy this property. Still, a large, useful class of colimits do satisfy it.

Working entirely in $\mathsf{Univ} + \mathsf{Pushouts}$, we characterize which colimits are preserved by pointed function types. These colimits are taken over pointed graphs with contractible quotients. Such colimits include, for example, pushouts, sequential colimits, and wedge sums. Specifically, consider a graph Γ with basepoint j_0, a diagram F over Γ, and a cocone (K, ι_K, κ_K) under F. For each vertex i, suppose that $F(i)$ has a basepoint b_i. Further, suppose that all maps in F are pointed. Provided that Γ is a tree (i.e., has a contractible quotient), we equip K with the structure of a cocone under F in $\mathcal{U}^*$, or the structure of a *pointed* cocone under F. In this case, we prove that for every pointed type P, the post-composition map

$$((K, \iota_{j_0}(b_{j_0})) \to_* P) \xrightarrow{e^{j_0}_{K,P}} \mathsf{lim}_{i:\Gamma^{\mathrm{op}}}(F(i) \to_* P)$$

is an equivalence for the canonical cocone $(\mathsf{colim}_\Gamma(F), \iota, \kappa)$ under F. Moreover, we prove that if $e^{j_0}_{K,P}$ is an equivalence for all P, then K and $\mathsf{colim}_\Gamma(F)$ are equal as pointed cocones. In this sense, the forgetful functor from pointed types to types creates colimits over trees.[1]

We also prove the converse: For every graph Γ, if the forgetful functor creates all colimits over Γ, then Γ is a tree. Therefore, for every pointed graph Γ, the following types are equivalent.

- Γ is a tree.
- The forgetful functor creates all colimits over Γ.

An Agda formalization of this equivalence is ongoing.

Finally, for *all* graphs Γ, the colimit of F over Γ in $\mathcal{U}^*$ can be defined directly as a non-recursive 2-HIT. By van Doorn et al. (2017), every such HIT has a construction in $\mathsf{Univ} + \mathsf{Pushouts}$. We discuss the possibility of recovering our equivalence from this general construction.

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[1] This result is already formalized for pushouts in the Agda-UniMath library.