On the logic of sets in the simplicial model*

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In the simplicial model [**?**], two competing notions of sets are available: the *discrete sets*, arising from the category **Set** of honest-to-goodness sets, and *h-sets* which are homotopical in nature and emerge from the interpretation of HoTT in **sSet**. Given the concrete nature of simplicial sets, one might expect that it would be possible to manually check whether a proposition involving h-sets is true or not in a explicit fashion.

Essentially, that intuition turns out to be correct, although some groundwork is required to facilitate the process. Indeed, the note [**?**] shows that already some attention to details is required when deriving the law of excluded middle in the simplicial model. In this talk we present a general method which streamlines the ideas of said note and simplifies the task at hand: we show that the truth of any reasonable proposition involving h-sets can be deduced by restricting to discrete sets.

The proof is divided in a few steps.

As we know, every h-set is homotopy equivalent to a discrete one; we show that this is still the case over an arbitrary context and that moreover the inclusion of discrete sets in h-sets is an equivalence in a type-theoretic sense.

More is true, if $\mathbf{T}$ is a reasonable type theory capable of expressing intuitionistic higher-order logic then the interpretations of $\mathbf{T}$ in discrete sets and in h-sets (which relies on the interpretation of $\mathbf{T}$ in HoTT) commute with the inclusion of the former in the latter.

Finally, since every simplicial set can be covered by a discrete one (its set of vertices), the above two facts allow us to transfer properties true of discrete sets to h-sets relatively directly.

More precisely, by restricting the type fibration of the CwA $(\mathbf{sSet}, \mathrm{Ty})$ to h-sets, we obtain a new CwA $(\mathbf{sSet}, \mathrm{Ty}_{\text{h-set}})$ which we can compare with the CwA of discrete sets $(\mathbf{sSet}, \mathrm{Ty}_{\text{disc}})$ as in the diagram below.

$$(\mathbf{Set}, \mathrm{Fam}) \xhookrightarrow{\cong} (\mathbf{sSet}, \mathrm{Ty}_{\text{disc}}) \xhookrightarrow{\simeq} (\mathbf{sSet}, \mathrm{Ty}_{\text{h-set}}) \longrightarrow (\mathbf{sSet}, \mathrm{Ty})$$

We show that the inclusion of discrete sets into h-sets above is a *local equivalence* in the sense of [**?**], meaning that it induces equivalences of contextual categories between each contextual slice, which in turns mean that these maps weakly reflect types and terms. Moreover, the left-most map from the usual CwA of sets and families of sets to $(\mathbf{sSet}, \mathrm{Ty}_{\text{disc}})$ is a *local isomorphism.*

We also show that the maps above commute, up to type-theoretic equivalence, with the various type constructors needed to encode intuitionistic higher-order logic in HoTT. As an intermediate step, we provide an interpretation of propositional truncation in the simplicial model based on image factorisation, which thus yields strict propositions.

Thus far, we would only be able to transfer properties of discrete sets to h-sets over set-based contexts. We obtain the general case by observing that, for any simplicial set Γ, the inclusion

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of vertices $\Gamma_0 \to \Gamma$ is a (-1)-connected map. Now, given any h-set-based proposition P over Γ we can pull it back to Γ_0, get a section there, then extend that section over the whole of Γ via a well-chosen lifting problem.

A portion of the present work was carried out as part of the author's master thesis, under the supervision of Peter LeFanu Lumsdaine, while the rest results from more recent developments. We also note that an effort to generalise the above to a broader class of models, such as $(\infty, 1)$-toposes, is currently work in progress.