Hilton-Milnor's theorem in ∞-topoi

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Hilton-Milnor's theorem is an important result in homotopy theory which provides a decomposition of the loop space on a finite wedge of spheres into a (weak) infinite product of spheres of varying dimensions. One of many consequences of this beautiful theorem is that the homotopy groups of spheres $\pi_j(S^i)$ generate all *homotopy operations*

$$\pi_{k_1}(-) \times \cdots \times \pi_{k_n}(-) \longrightarrow \pi_k(-)$$

under the Whitehead bracket. Such operations are akin to Steenrod operations on cohomology. In its most general form, the theorem states that if $X_1, \cdots, X_n$ are connected pointed spaces, there is a natural homotopy equivalence

$$\Omega\Sigma(X_1 \vee \cdots \vee X_n) \;\simeq\; \widetilde{\prod_w}\, \Omega\Sigma w(X_1, \cdots, X_n) \qquad (*)$$

where $\widetilde{\prod}$ denotes a weak product, which ranges over the words w in any choice of basis for the *free Lie ring* on n generators. We show that this statement holds in any hypercomplete ∞-category with universal pushouts.

The proof, whose essence closely parallels classical accounts like [Whi78], becomes purely formal and conceptually simpler. It uses only basic ∞-categorical constructions, in a similar spirit to [SKD21]. If hypercompleteness is assumed, the argument becomes completely internal and holds in any ∞-category with universal pushouts, examples of which include *elementary* ∞-topoi, as defined and studied in [Ras18]. Moreover, we believe that this proof could be adapted in intensional type theory, upon imposing a hypercompleteness axiom.

In an ∞-topos, the loop functor Ω is part of an equivalence between connected pointed objects and group objects (group-like internal E_1-algebras). Through this equivalence $\Omega\Sigma X$ can be identified with the *free group* on a pointed object X. The right hand side of $(*)$ is a homotopy coherent analogue of the associated graded $\mathrm{gr}(\Gamma_* G)$ of the lower central series $\{\Gamma_i G\}$ of a group G. Hilton-Milnor's theorem can thus be read as a structure result on (objects underlying) free groups on finite connected wedge sums.

In this talk, after providing some context for Hilton-Milnor's theorem, we present the main ideas of its ∞-categorical proof. We will explain why hypercompleteness should be assumed to obtain a purely *internal* proof. If time permits, we will discuss some work in progress on applications of this result to the theory of higher groups.

References

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