Talk Proposal for HoTT: Goursat Homology *

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Abstract. The talk will be an introduction to a theory of homology that arises in the model theory of modules, where positive primitive formulae are prevalent. The definition of the boundary map is based on a theorem about groups due to Goursat. It presents an issue as to just what kind of homology this is. It has features of cubical homology, but also may be seen as a kind of a double cover of simplicial homology.

Goursat's Theorem. Let G and H be groups and consider a subgroup $\Gamma \leq G \times H$. Denote by $\pi_G(\Gamma) := \{g \in G \mid \exists h \in H \ ((g,h) \in \Gamma)\}$ and $\pi_H = \{h \in H \mid \exists g \in G \ ((g,h) \in \Gamma)\}$ the respective projections of Γ onto the component groups. Goursat [1] proved that there is a canonical isomorphism

$$\frac{\pi_G(\Gamma)}{\Gamma \cap (G \times 1)} \cong \frac{\pi_H(\Gamma)}{\Gamma \cap (1 \times H)},$$

whose graph is induced by Γ.

Positive Primitive Formulae. Let $\mathcal{L} = \mathcal{L}(\sigma)$ be the first order-language based on a signature $\sigma = (+, -, 0, ...)$ that extends that of abelian groups. An abelian $\mathcal{L}$-structure $(A, +, -, 0, ...)$ is an abelian group (interpreted by the shown symbols of σ) in which all the function and predicate symbols of σ respect the underlying abelian group structure. Precisely, the function symbols interpret homomorphisms of the underlying abelian groups and the predicates interpret subgroups. A positive primitive formula of $\mathcal{L}$ is an existentially quantified conjunction of atomic formulae. For example, in the language $\mathcal{L} = \mathcal{L}((+, -, 0) \cup R)$ of left R-modules (where every element $r \in R$ serves as a unary function symbol), an atomic formula is equivalent to a linear equation

$$r_1 u_1 + r_2 u_2 + \cdots + r_n u_n \doteq 0$$

so that a positive primitive formula is nothing more than an existentially quantified system of linear equations. A positive primitive formula $\varphi(u_1, u_2, \ldots, u_n)$ in the free variables shown defines in an abelian $\mathcal{L}$-structure A a subgroup $\varphi(A) \leq A^n$ and this definition is natural in the sense that it defines a subfunctor of the n-th power of the forgetful functor from the category of abelian $\mathcal{L}$-structures to abelian groups. We identify two positive primitive formulae $\varphi(u_1, \ldots, u_n)$ and $\psi(u_1, \ldots, u_n)$ if they define in every abelian $\mathcal{L}$-structure the same group, that is, if they yield the same subfunctors of the n-th power of the forgetful functor.

Quantification. There are two quantifiers that preserve the positive primitive property of a formula: the existential quantifier and instantiation by the constant 0. So if $\varphi(u, v)$ is a positive primitive formula in two free variables, then $\exists v\, \varphi(u, v)$, $\exists u\, \varphi(u, v)$, $\varphi(0, v)$, and $\varphi(u, 0)$ are positive primitive formulae in one free variable. Using this idiom, Goursat's Theorem may be expressed as an isomorphism of sorts

$$\frac{\exists v\, (\varphi(u, v))}{\varphi(u, 0)} \cong \frac{\exists u\, (\varphi(u, v))}{\varphi(0, v).}$$

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Definition of Homology. A complex $C. = C.(\mathcal{L})$ of abelian groups is defined in [2] whose n-chains C_n are formal integral sums of positive primitive formulae in the free variables $(u_0, u_1, \ldots, u_n)$. Thus C_n is the free abelian group on formal symbols $[\varphi(u_0, u_1, \ldots, u_n)]$ associated to equivalence classes of positive primitive formulae. The boundary map is defined as in reduced cubical homology [3]:

$$d_n[\varphi] := \sum_{i=0}^{n} (-1)^i \left([\exists u\, \varphi(u_0, \ldots, u_{i-1}, u, u_i, \ldots, u_{n-1})] - [\varphi(u_0, \ldots, u_{i-1}, 0, u_i, \ldots, u_{n-1})]\right).$$

Furthermore, $C_{-1} := \mathbf{Z}$ with $d_0 \colon C_0 \to \mathbf{Z}$ is given by the augmentation map $[\varphi] \mapsto 1$, and $C_n = 0$ for $n < -1$.

For a positive primitive formula $\varphi(u, v)$, considered as a 1-chain, the boundary is given by

$$d_1[\varphi(u_0, u_1)] = [\exists u\, \varphi(u, u_0)] - [\varphi(0, u_0)] - ([\exists v \varphi(u_0, v)] - [\varphi(u_0, 0)]),$$

which are the relations in C_0 given by Goursat's Theorem and give the homology its name. Using this property, one proves [2] that when $\mathcal{L}$ is the language for left R-modules, the 0-dimensional homology of $C.$ is isomorphic to $K_0(\mathrm{Ab}(R))$, the Grothendieck group (modulo short exact sequences) of the free abelian category over R (considered as a one-object preadditive category).

Degeneracy Maps. The two quantifiers provide a front and back face in the alternating sum of the boundary map, which suggests that the homology is akin to cubical homology. On the other hand, each quantifier comes with its own degeneracy map, in the sense that the usual adjoint relation is satisfied. Precisely, to every positive primitive formula $\varphi(u_0, \ldots, u_{n-1})$ in n free, there are two kinds of degeneracy map:

$$D_i^+[\varphi] = [\varphi(u_0, \ldots, u_{i-1}, u_i, \ldots, u_n)] \text{ and } D_i^-[\varphi] = [\varphi(u_0, \ldots, u_{i-1}, u_i, \ldots, u_n) \wedge u_i \doteq 0].$$

It is well known [4, §I.9] that D_i^+ is the right adjoint of existential quantification at the i-th variable, but there is also a dual property that holds. Namely, the D_i^- is the left adjoint of the "instantiation of 0 for the i-th variable" quantifier. These two quantifiers, together with their associated degeneracy maps, could also be used to obtain two theories of simplicial homology, but neither with a ready interpretation. From this point of view, Goursat Homology seems to be built up using two dual simplicial homologies.

References

[1] Goursat, M.E., Sur les Substitutions Orthogonales et les Divisions Réguliers de L'Éspace, Annales Scientifiques de l'École Normale Supérieure **6**, Troisième Série 1889, pp. 99-102.

[2] Herzog, I, Linear Algebra over a Ring, arXiv:0909.0436, September 2009.

[3] Massey, W.S., *Singular Homology Theory,* Graduate Texts in Mathematics **70**, Springer-Verlag, 1980.

[4] Mac Lane, S. and Moerdijk, I., *Sheaves in Geometry and Logic; A First Introduction to Topos Theory,* Universitext, Verlag, 1992.