Directed Types are Weak ω-Categories[*]

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Both Lumsdaine [3] and van den Berg and Garner [5] have shown that the tower of identity types associated to each type in a dependent type theory with identity types has the structure of a Batanin-Leinster weak ω-category (see [1, 2]). In this paper we seek to understand and generalise this phenomenon. Our formalism revolves around algebraic objects called *globular multicategories*, and is sufficiently general to include *directed* analogues of Garner and van den Berg's results.

As input to our construction, we require a model of dependent type theory with identity types in the form of a triple $(\mathcal{C}, \mathcal{F}, \mathcal{R})$ where $\mathcal{C}$ is a category, and $\mathcal{F}$ is a collection of (higher) spans in $\mathcal{C}$ called *two-sided fibrations*, and $\mathcal{R}$ is a collection of transformations between these spans called *representors*. (In the undirected case, representors are typically weak equivalences.) These data can be obtained starting from an identity type category in the sense of [5], or a collection of two-sided factorisations in the sense of North [4].

Given this input, there are two steps, which mirror those taken by van den Berg and Garner. First, we construct an intermediate globular multicategory $\mathrm{Span}(\mathcal{C}, \mathcal{F})$. In general, a globular multicategory consists of a globular set of *types*, and a collection of arrows between types, called *terms*. In this case, types are the spans in $\mathcal{F}$, and terms are natural transformations between these spans. The (directed) identity types in our starting type theory always induce extra structure on the globular multicategory $\mathrm{Span}(\mathcal{C}, \mathcal{F})$; we refer to globular multicategories with this structure as *globular multicategories with homomorphism types*. As a second step, we show that the globular multicategory with homomorphism types freely generated from a type is a normalised contractible globular operad. Consequently, each type in a globular multicategory with homomorphism types carries the structure of a a weak ω-category in the sense of Batanin and Leinster. Thus, each type in our initial type theory induces such a weak ω-category.

$$\mathrm{TypeTheory} \longrightarrow \mathrm{GlobularMulticategory} \longrightarrow \omega\text{-}\mathrm{Category}$$

References

[1] Michael Batanin. Monoidal globular categories as a natural environment for the theory of weak n-categories. *Advances in Mathematics*, 136(1):39 – 103, 1998.

[2] T. Leinster. *Higher Operads, Higher Categories*. Cambridge, UK: Cambridge University Press, August 2004.

[3] Peter LeFanu Lumsdaine. Weak omega-categories from intensional type theory. *Logical Methods in Computer Science*, Volume 6, Issue 3, Sep 2010.

[4] Paige Randall North. Two-sided weak factorization systems for directed type theory. 2019.

[5] Benno van den Berg and Richard Garner. Types are weak ω-groupoids. *Proceedings of the London Mathematical Society*, 102(2):370–394, 2011.