Theory and Implmentation of Bicubical Directed Type Theory

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Riehl and Shulman introduced a simplicial type theory for synthetic $(\infty,1)$-category theory, as an extension of homotopy type theory with extra judgemental structure for representing simplicial shapes. While the “standard” (non-constructive) model of HoTT is in simplicial sets, the “standard” model of this simplicial type theory is in bisimplicial sets. Riehl and Shulman define some first constructions in this setting, e.g. defining $(\infty,1)$-categories as Segal and Rezk types and defining co- and contravariant fibrations of groupoids. Buchholtz and Weinberger have defined (co)Cartesian fibrations and shown that this type theory supports some substantial applications. Cavallo, Riehl, and Sattler proved a directed analog of the univalence axiom for a universe of covariant fibrations of groupoids, stating that the morphisms in this universe are exactly the functions between types.

In previous work, we developed a bicubical variant of this simplicial type theory, which gives a constructive model of this directed univalence principle. First, we implement the underlying homotopy type theory using a cubical type theory, to give a constructive interpretation to the base system. Next, we replace the simplicial shapes used to talk about categories with cubical ones, using the “Dedekind” cube category (Cartesian cubes with connections but no reversals), so that we can constructively build a universe of covariant fibrations of groupoids using tininess of the cubical interval. Using the “glue” types from cubical type theory, we show that functions determine morphisms in this universe. Finally, using the cobar construction following Coquand, Ruch, and Sattler, we show that morphisms in the universe of covariant fibrations are exactly functions.

In this talk, we will describe an in-progress prototype implementation of this bicubical directed type theory, as an extension of the `cooltt` proof assistant developed by Angiuli, Cavallo, Favonia, Harper, Mullanix, and Sterling. We will also describe some new theoretical tricks we have developed in service of this implementation. To sidestep the complexities of implementing the connection operations of the Dedekind cube category, we (in joint work with Robert Rose) have given a generalized definition of covariant fibrations, using a variation of the idea of “generic points” used in the diagonal Kan operation of Cartesian cubical type theory; this definition works for any cube category that supports an inequality cofibration. To define this and related filling problems while staying entirely within Kan types, we (in joint work with Reed Mullanix) have generalized the fibrant extension types introduced by Riehl and Shulman: by adding syntax for a new context zone for fiberwise fibrant variables, extension types can be separated into several smaller type constructors that are each Kan in isolation. Our prototype implementation currently supports these aspects, but not yet the universe of covariant fibrations with directed univalence.