Propositional dependent type theories: a conservativity result for homotopy elementary types*

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In recent years, there has been a growing interest in various weakenings for theories of dependent types, particularly those weakenings with respect to the strength of the computation rules of the type constructors. In detail, when a dependent type theory has a type constructor that relies on a propositional equality instead of a judgemental equality, as is the case in Martin-Löf Type Theory, one says that the type constructor is in *propositional* form. Thus, a dependent type theory will have e.g. *propositional* identity types if it is endowed with a type constructor satisfying the usual rules of intensional identity types, except for the judgemental equality of its computation rule: whenever we are given judgements $x, y : A;\ p : x = y \vdash C(x, y, p) : \text{Type}$ and $x : A \vdash q(x) : C(x, x, r(x))$, in place of asking that the judgement $x : A \vdash J(x, x, r(x), q) \equiv q(x)$ holds -here J denotes the identity type eliminator-, we only ask that a judgement:

$$x : A \vdash H(x, q) : J(x, x, r(x), q) = q(x)$$

holds; see [3, 4] for more details.

Coquand and Danielsson [3] were the first to consider propositional identity types. This type constructor has since been extensively studied by van den Berg [4], who proposed and analysed a notion of semantics of dependent type theories with propositional identity types, based on the notion of the *path category*. Additionally, Bocquet [5] examines the property of *Morita equivalence* between such a type theory and its extension obtained by strictifying the computation rule for identity types. One might consider the same form of weakening for the computation rule of dependent sum types and dependent product types: these type constructors satisfying an h-propositional computation rule will be called *propositional dependent sum types* and *propositional dependent product types* respectively.

In this talk, we will discuss a dependent type theory that includes propositional identity types, propositional dependent sum types, and propositional dependent product types, along with an arbitrary family of basic types. We will refer to such a theory as a *propositional type theory*. A recent paper [6] has demonstrated that, in such propositional type theories, it is possible to decide the derivability of a term-judgment in quadratic time. The authors of this paper also conjecture that such a theory is sufficient for performing all of constructive mathematics and formalising most of the HoTT book.

The aim of this talk is to address this question, by identifying a particular family of type-judgements, called *h-elementary*, of a propositional type theory such that the corresponding extensional type theory is conservative over it *relative* to the family of h-elementary types. In detail, if $| \cdot |$ is the canonical interpretation of the propositional type theory into the corresponding extensional strictification, we show using insights from homotopy type theory that, whenever the extensional type theory infers a judgement of the form $|\Gamma| \vdash t : |T|$ for some h-elementary judgement $\Gamma \vdash T : \text{Type}$ of the propositional type theory, then the latter infers a judgement of the form:

$$\Gamma \vdash \tilde{t} : T$$

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and the former infers the judgement $|\Gamma| \vdash |\tilde{t}| \equiv t$. The h-elementary type-judgements of the propositional type theory happen to be the ones obtained by inductively applying the type formation rules to the basic h-sets. Hence, our main result informally asserts that, for judgements essentially concerning h-sets, reasoning with extensional or propositional type theories is equivalent.

Therefore, despite this non-negligible weakening of the propositional type theory with respect to the extensional one, nonetheless there is actually an interesting family of *statements* where the two theories have the same deductive power. Our argument is fully presented in [7]. It is obtained by adapting a proof strategy introduced in [2] and exploits the notion of category with attributes [1] to phrase the semantics of theories of dependent type.

In detail, we recursively define a family of *canonical homotopy equivalences* between the h-elementary contexts of the given propositional type theory, resulting in a quotient modulo these equivalences that happens to model the extensional type theory generated by the basic h-sets of the propositional one. The key feature that this model enjoys, enabling us deduce the conservativity property, is that the interpretation of propositional type theory in it is surjective with respect to context-, type-, and term-judgements. Our proof also relies on a characterisation, in terms of two *projection* rules, two propositional *β-reduction* rules, and a propositional *η-expansion* rule, for the usual elimination rule and the propositional computation rule of the propositional dependent sum types in presence of propositional identity types. This characterisation is analogous to that of dependent sum types in extensional type theory. In spite of the amount of work in the purely syntactic part of this research, namely in the analysis of the family of canonical equivalences and in the characterisation of the rules of propositional dependent sum types, we underline the central and fundamental role of the soundness property of the semantics induced by the class of categories with attributes.

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