Commuting Cohesions*

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Shulman's spatial type theory [13] internalizes the modalities of Lawvere's axiomatic cohesion [3] in a homotopy type theory, enabling many of the constructions from Schreiber's modal approach to differential cohomology [12] to be carried out synthetically. In spatial type theory, every type carries a spatial cohesion among its points and every function is continuous with respect to this. But in mathematical practice, objects may be spatial in more than one way at the same time; for example, a simplicial space has both topological and simplicial structures.

It is a classical theorem that the homotopy type of a manifold may be computed as the realization of a (topologically discrete) simplicial set associated to the Čech nerve of a good open cover of the manifold. This theorem relates a simplicial set to a continuous space, via an intermediary simplicial space which is both continuous and simplicial at the same time — the Čech nerve of the cover. But in spatial homotopy type theory there is only one notion of crisp variable, and therefore just access to only one sort of spatiality.

In our paper [8] — the subject of this talk — we put forward a modification of spatial type theory to allow for multiple axes of spatiality. Our theory works by allowing for a meet semi-lattice of '*focuses*', each with a separate notion of crisp variable and pair of adjoint (co)modalities $\flat$ and $\sharp$. Like spatial type theory, our custom type theory gets us to the coalface of synthetic homotopy theory very efficiently while staying simple enough to be used in an informal style.

The presence of multiple notions of crispness forces a more complex context structure than spatial type theory's separation of the context into a crisp zone and cohesive zone. Similar to other modal dependent type theories [4, 2, 9], we annotate each variable with modal information, here, the focuses for which that variable is crisp. The typing rules for the modalities of each focus then work essentially independently.

In addition to allowing us to formalize the theorem about Čech nerves of open covers, our type theory will be able to handle the equivariant differential cohesion used by Sati and Schreiber in their *Proper Orbifold Cohomology* [11], as well as the nested focuses of Schreiber's supergeometric *solid* cohesion [12]. This extends the work of Cherubini [1] and the first author [5, 6, 7] of giving synthetic accounts of the constructions of Schreiber [12] and Sati-Schreiber [11].

Positing an additional focus does not disturb arguments made using existing focuses, so we also expect our theory to be helpful for dipping into simplicial arguments in the course of other reasoning by adding a simplicial focus and making use of the new modalities. The problem of defining simplicial types in ordinary Book HoTT remains open, and there are now a number of different approaches to constructing simplicial types which each use some extension to the underlying type theory. In this paper, we will use the simplicial cohesive modalities to define the Čech nerve of a map and the realization or colimit of a simplicial type. We believe

*Supported by Tamkeen under the NYU Abu Dhabi Research Institute grant CG008.
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our approach would pair nicely with other approaches to simplicial types for the purposes of synthetic $(\infty, 1)$-category theory such as [10], where the sk_0 modality would take the core of a Rezk type.

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