The unifying power of modal type theory

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A decade ago, the book "Homotopy Type Theory" brought a new and fast-growing subject to a wider audience, using a specific formal system now known as "Book HoTT": intensional Martin-Löf type theory with axioms for univalence and higher inductive types. In the years since then, much progress has been made using Book HoTT and its semantics in higher categories. But in addition, a wide array of other formal systems have arisen: cubical type theory, two-level type theory, cohesive type theory, simplicial type theory, spectral type theory, directed type theories, guarded type theories, parametric type theories, indexed type theories, higher observational type theory, and so on.

Despite the varied nature of these new theories, there is a common thread running through them all. *Modal* homotopy type theory is a general family of theories that parametrize types by "modes", and introduce type-formers called "modalities" that take a type of one mode to a type of a possibly-different mode. Its intended semantics lies in *diagrams* of higher categories indexed by a 2-category, with modalities interpreted by functors. Many of the above type theories are modal type theories by definition; but in fact, all of them can be viewed as modal type theories, perhaps with additional structure. At an intuitive level, this is not hard to see: in each case we add some new type-forming operations, which can therefore be viewed as modalities. But making it as precise as possible is an active research area, which has seen substantial progress in the past few years.

In particular, there are now candidate *general* modal dependent type theories that can be specified once and for all, with the mode 2-category as a parameter. Two such theories are MTT [GKNB21], with "positive" modalities, and FitchTT [GCK$^+$22], with "negative" modalities. This raises the possibility that the wide range of successor systems to Book HoTT could all be obtained as instances of such a general theory, rather than needing to be specified by hand and studied one by one; and that therefore a single proof assistant implementing such a theory could be used to formalize all of them.

However, there are obstacles to realizing such a vision. Syntactically, many features of particular modal type theories have not yet been given a general treatment, such as dependent modalities and cofibrations, computational modalities, modal inductive and coinductive types, and computational context locks/divisions. And semantically, general modal type theories have not yet been shown to have models in general diagrams of (higher) categories; the straightforward semantics requires all the modality functors to have extra left adjoints ("locks") that are not visible in the syntax.

In [Shu23] I proposed a partial solution to the latter semantic problem. Namely, given any diagram of sufficiently limit-preserving functors, we can construct a new diagram, called its *co-dextrification*, in which all the functors do have left adjoints. We can regard this as a sort of "coherence theorem" that turns a naturally-occurring categorical structure into a stricter version that corresponds more directly to type theory. The "contexts" in the co-dextrification can be thought of as remembering just enough about how they were built up from modal types that the lock functors can be defined. But the *types* in the co-dextrification are the same as

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those in the original diagram, so interpreting a general modal type theory in the former still yields results about the latter.

After describing general modal type theories and the co-dextrification, I will discuss what has to be added to these theories to recover some of the recent type theories mentioned above, touching on unpublished work in progress with Thorsten Altenkirch and Ambrus Kaposi, Astra Kolomatskaia, and Daniel Gratzer. These examples suggest ways that general modal type theories could be enhanced, hopefully bringing us closer to a reunification of the discipline under the banner of modal type theory.

However, even with these enhancements remaining for the future, the modal perspective already provides a common *conceptual* framework, as well as systematic ways to *combine* multiple innovative type theories into a single system. Thus, general modal type theories suggest a path to the implementation of more generic, flexible, and modular proof assistants.

References

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