Opetopic Methods in Homotopy Type Theory

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Joint work with Eric Finster

Introduction

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Problem: leads to a situation of circular dependency in HoTT where one would have to define these structures coherently in the first place.

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Examples: the universe of types, the opetopic type associated to a type, adjunctions, joins, …

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Most of our work has been formalised in Agda using postulates and rewrite rules to define the universe of polynomial monads.

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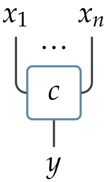
Elements depicted as corollas:



Polynomial monads

A cartesian polynomial monad is a polynomial functor along with a unit η and a multiplication μ:

$$\eta_M : (i : \mathsf{Idx}_M) \to \mathsf{Cns}_M(i)$$

$$\mu_M : \{i : \mathsf{Idx}_M\}\ (c : \mathsf{Cns}_M(i)) \to \overrightarrow{\mathsf{Cns}_M}(c) \to \mathsf{Cns}_M(i)$$

Notation. For any monad $M : \mathcal{M}$,

$$\mathsf{Fam}_M :\equiv \mathsf{Idx}_M \to \mathcal{U}$$

For any type family $X : \mathsf{Fam}_M$,

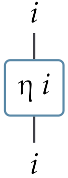
$$\overrightarrow{X}(c) :\equiv (p : \mathsf{Pos}_M(c)) \to X(\mathsf{Typ}_M(c, p))$$

Polynomial monads

The unit

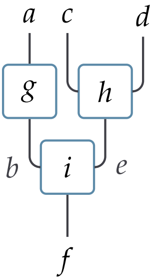
$$\eta_M : (i : \mathsf{Idx}_M) \to \mathsf{Cns}_M(i)$$

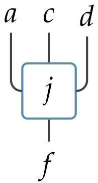
Units $\eta(i)$ are *unary* constructors whose source and target have the same sort:



$$\mu_M : \{i : \mathsf{Idx}_M\}\ (c : \mathsf{Cns}_M(i)) \to \overrightarrow{\mathsf{Cns}_M}(c) \to \mathsf{Cns}_M(i)$$

The multiplication "contracts" a tree of constructors while preserving the type of positions and their sort.





The operation μ_M is associative and unital with units η_M:

$$\mu_M(c, \lambda\, p \to \eta_M(\mathsf{Typ}_M(c, p))) \equiv c$$
$$\mu_M(\eta_M(i), d) \equiv d(\eta\text{-}\mathsf{pos}\,(i))$$
$$\mu_M(\mu_M(c, d), e) \equiv \mu_M(c, (\lambda\, p \to \mu_M(d(p), (\lambda\, q \to e(\mathsf{pair}^{\mu}(p, q))))))$$

Identity monad

We populate the universe by introducing codes for our monads and by defining the relevant decoding functions.
The identity monad $\mathsf{Id} : \mathcal{M}$ has a single unary constructor.

$$
\begin{aligned}
\mathsf{Idx}_{\mathsf{Id}} &:\equiv \mathbf{1} \\
\mathsf{Cns}_{\mathsf{Id}}(i) &:\equiv \mathbf{1} \\
\mathsf{Pos}_{\mathsf{Id}}(c) &:\equiv \mathbf{1} \\
\mathsf{Typ}_{\mathsf{Id}}(c, p) &:\equiv *
\end{aligned}
$$

The monad structure is trivial.

Baez-Dolan slice construction

For any monad $M : \mathcal{M}$ and family $X : \mathsf{Fam}_M$, their slice construction is the monad $\boxed{M/X : \mathcal{M}}$.

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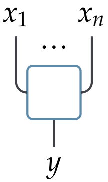
Indices are *frames*: quadruplets $(i, y) \triangleleft (c, x)$ representing a constructor of M whose sources and target are decorated with elements in X.

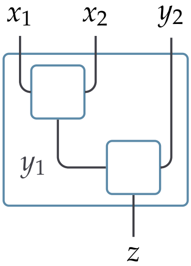


Baez-Dolan slice construction

Constructors are well-founded trees of frames which multiply to their indexing frame under the operation μ_M. Defined as an inductive types whose constructors are:

$\mathsf{lf}(i, x)$

$\mathsf{nd}((i, z) \triangleleft (c, y), t)$



Baez-Dolan slice construction

The positions of a constructor are *paths* in the tree it represents from its root to its different nodes:

$$\begin{aligned}\mathsf{Pos}_{M/X}(\mathsf{lf}(i,x)) &:\equiv \mathbf{0}\\ \mathsf{Pos}_{M/X}(\mathsf{nd}((i,z) \triangleleft (c,y),t)) &:\equiv \mathbf{1} + (p : \mathsf{Pos}_M(c)) \times \mathsf{Pos}_{M/X}(t_p)\end{aligned}$$

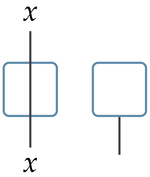
The typing function projects out the constructor associated to a node at a specified position:

$$\begin{aligned}\mathsf{Typ}_{M/X}(\mathsf{nd}((i,z) \triangleleft (c,y),t),\mathsf{inl}(*)) &:\equiv (i,z) \triangleleft (c,y)\\ \mathsf{Typ}_{M/X}(\mathsf{nd}((i,z) \triangleleft (c,y),t),\mathsf{inr}(p,q)) &:\equiv \mathsf{Typ}_{M/X}(t_p,q)\end{aligned}$$

Baez-Dolan slice construction

The constructors of M/X illustrated.

$\mathsf{lf}(i, x)$



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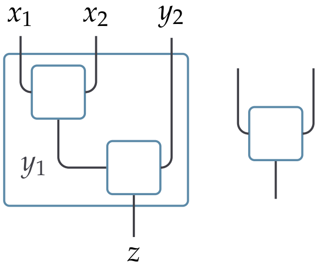

Algebras

An algebra for a monad M is

- a family $X_0 : \mathsf{Fam}_M$,
- a family $X_1 : \mathsf{Fam}_{M/X_0}$.

such that the type

$$(y : X_0(i)) \times X_1((i, y) \triangleleft (c, x))$$

is contractible for any constructor $c : \mathsf{Cns}_M(i)$ and values $x : \overrightarrow{X_0}(c)$.

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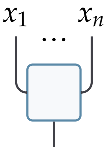

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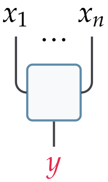

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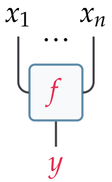

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$\mathcal{O}_M$ is the type of M-opetopic types.

Opetopic types

Examples of opetopic types:

- $\infty\text{-Grp} = (X : \mathcal{O}_{\mathsf{Id}}) \times \text{is-fibrant}(X)$
- $(\infty, 1)\text{-Cat} = (X : \mathcal{O}_{\mathsf{Id}}) \times \text{is-fibrant}(X_{>0})$

The universe

0-cells

Types and their fibrant relations assemble into the $(\infty, 1)$-category

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Its family of objects $\mathcal{U}^o_0$ is the universe of types $\mathcal{U}$:

$$\mathcal{U}^o_0(*) \equiv \mathcal{U}$$

The universe

1-cells

The family of 1-cells

$$\mathcal{U}_1^o : \mathsf{Id} \times_{\mathsf{Id}/\mathcal{U}_0^o} \to \mathcal{U}$$

is a binary relation on $\mathcal{U}$.

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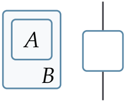
For example,

$\mathcal{U}_1^o($ 

$) \simeq (R : (a : A)\,(b : B) \to \mathcal{U}) \times \mathsf{is\text{-}fibrant}(R)$

The family of 2-cells

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relates a *source* pasting diagram of 1-cells to a *target* 1-cell.

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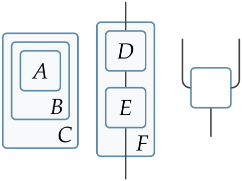
$\mathcal{U}_2^o($ 

$)$

$$\simeq (R : (a : A)\,(b : B)\,(c : C) \to (d : D(a,b))\,(e : E(b,c))\,(f : F(a,c)) \to \mathcal{U}) \times \text{is-fibrant}(R)$$

Formally, the domain of our relations are frames of the universal fibration $\mathcal{U}^o_\bullet \to \mathcal{U}^o$.

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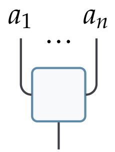

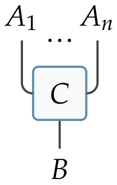

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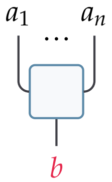

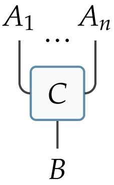

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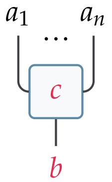

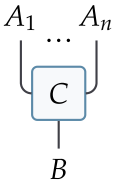

Thank you for your attention.